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Super Sudoku Squares

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Abstract

We refer to a completed Sudoku puzzle as a *Sudoku Square of order 3*. We define a more restricted, but natural, extension — a *Super Sudoku Square of order n*, and determine for which orders n one exists.

1 Introduction

In recent years, sudoku puzzles have become extremely popular. The modern day sudoku puzzle first appeared in 1979 as a puzzle called "Number Place" in Dell Magazine [6]. They were designed by Howard Garns [2], a freelance puzzle constructor and retired architect. A *sudoku puzzle* is a 9×9 square grid in which every cell contains exactly one *symbol* (typically denoted with integers 1 through 9) in such a way that each 1×9 row contains each symbol exactly once, each 9×1 column contains each symbol exactly once, and each 3×3 sub-square tiling the grid starting at the top left (often called *blocks* or *regions*) contains each symbol exactly once. An example of a completed Sudoku is presented in Figure 1.

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1	6	8	5	7	3	9	2	4
2	4	9	6	8	1	7	3	5
3	5	7	4	9	2	8	1	6
4	9	2	8	1	6	3	5	7
5	7	3	9	2	4	1	6	8
6	8	1	7	3	5	2	4	9
7	3	5	2	4	9	6	8	1
8	1	6	3	5	7	4	9	2
9	2	4	1	6	8	5	7	3

Figure 1:

Generalizations of Sudoku puzzles have been popular over the years. The most closely related to this topic are Retransmission Permutation Arrays. The existence of such designs have been studied by people such as Wanless and Zhang in [13] as well as Dinitz, Paterson, Stinson, and Wei in [5]. Retransmission Permutation Arrays are $n \times n$ grids used to resolve problems in overlapping channel transmissions.

There has also been a great deal of literature written on orthogonal arrays based on Sudoku puzzles such as in [8], which deals with Sudoku-like arrays, codes, and orthogonality. Other Sudoku arrays have been studied by Lorch in [10]. Samurai Sudoku-based space filling curves have also been a variation of Sudoku-arrays that has been used to pool data from multiple sources in [14].

Other generalizations of Sudoku puzzles such as Magic Sudoku variants have been studied in [1]. Some variations include Sudoku using partially ordered sets in [4], modular magic squares in [11], and strongly symmetric self-orthogonal diagonal Sudoku squares in [12]. Others have taken a different approach and defined (a, b)-Sudoku Latin squares and (a, b)-Sudoku Pair Latin squares for a given factor pair (a, b) as in [7].

We propose yet another way to extend the idea of the traditional Sudoku puzzle.

2 Main Results

We coordinatize this as follows. The rows are divided into three groups, called *fat rows*, (fr), labeled 1, 2, 3. Each fat row consists of three *skinny rows*, (sr), also labelled 1, 2, 3. Similarly, we have *fat columns*, (fc), and *skinny columns*, (sc), also similarly labelled. Each of the 81 occupied cells is assigned a label of 4 coordinates, $(\mathrm{fr}, \mathrm{sr}, \mathrm{fc}, \mathrm{sc})$. For example, in the above array, symbol 9 occurs in these 9 cells: $(1,1,3,1)$, $(1,2,1,3)$, $(1,3,2,2)$, $(2,1,1,2)$, $(2,2,2,1)$, $(2,3,3,3)$, $(3,1,2,3)$, $(3,2,3,2)$, $(3,3,1,1)$.

To be a valid Sudoku square, each row, and each column must contain each symbol exactly once. Furthermore, each *block* (= intersection of a fat row and a fat column) must contain each symbol exactly once. Heres the definition for values other than 3.

Let N be a finite non-empty set of size $|N| = n$. (We took $N = \{1,2,3\}$ above.) A *Sudoku square* of *order* n, $\mathrm{SS}(n)$, is a function $f : N^4 \to S$, where S is a set of n^2 symbols, satisfying the following properties. (Here $f(c)$ is the symbol occupying cell c.) Each of the following functions is a bijection from N^2 to S: for all $i, j \in N$,

$$\begin{aligned}(x,y) &\to f(i,j,x,y),\\(x,y) &\to f(x,y,i,j),\\(x,y) &\to f(i,x,j,y).\end{aligned}$$

To simplify notation, we denote these three properties by $f(i,j,\_,\_)$, $f(\_,\_,i,j)$, $f(i,\_,j,\_)$ respectively.

We leave the proof of the following to the reader:

Theorem 1. *There is a* $\mathrm{SS}(n)$ *for every positive integer* n.

The three properties above are defined by choosing two of the four blanks in $f(\ ,\ ,\ ,\ )$ to be replaced by underscores _. But there are six such choices. Accordingly, we define a *Super Sudoku square* of *order* n, $\mathrm{SSS}(n)$, to be a function $f : N^4 \to S$, satisfying, for all $i, j \in N$, all six of the properties

$$\begin{aligned}&f(i,j,\_,\_),\\&f(\_,\_,i,j),\\&f(i,\_,j,\_),\\&f(i,\_,\_,j),\\&f(\_,i,j,\_),\\&f(\_,i,\_,j).\end{aligned}$$

Do these exist? The SS(3) above is actually a SSS(3)! Let's check the last three properties at $i = 3$, $j = 2$.

	3			4			8	
	1			5			9	
	2			6			7	

Figure 2: $f(3, \_, \_, 2)$

			4	9	2			
			7	3	5			
			1	6	8			

Figure 3: $f(\_, 3, 2, \_)$

	5			9			1	
	8			3			4	
	2			6			7	

Figure 4: $f(\_, 3, \_, 2)$

Note that in all three cases, the nine occupied cells contain each of the nine symbols exactly once. Those are three of the twenty-seven properties; we leave the rest to the reader to check.

So for which positive integer n is there a SSS(n)? Try $n = 2$, and you'll fail. So 2 is out.

To investigate this question, we need an alternate definition of super sudoku squares. For this, we will take S to be $N \times N$ from now on, so our symbols are ordered pairs of elements of N. For a positive integer k, we say an n^k by k array is *complete* if its rows consist of the n^k vectors with k coordinates from N, each one occurring exactly once. We define A to be an n^4 by 6 *Super Sudoku array* of *order* n, SSA(n), if its six columns are indexed by the set $\{\text{fr}, \text{sr}, \text{fc}, \text{sc}, \text{s}, \text{t}\}$, and the n^4 by 4 subarray consisting of the columns of A indexed by S is complete, for each of the 7 following sets S: $\{\text{fr}, \text{sr}, \text{fc}, \text{sc}\}$, $\{\text{fr}, \text{sr}, \text{s}, \text{t}\}$, $\{\text{fc}, \text{sc}, \text{s}, \text{t}\}$, $\{\text{fr}, \text{fc}, \text{s}, \text{t}\}$, $\{\text{fr}, \text{sc}, \text{s}, \text{t}\}$, $\{\text{sr}, \text{fc}, \text{s}, \text{t}\}$, $\{\text{sr}, \text{sc}, \text{s}, \text{t}\}$.

Theorem 2. *There is a* SSS(n) *if and only if there is a* SSA(n).

Proof. First, suppose there is a SSS(n). We construct a SSA(n) as follows: for all $(i, j, k, l) \in N^4$, if cell (i, j, k, l) contains symbol (a, b), then (i, j, k, l, a, b) is a row of the SSA. The condition imposed by the first set S listed above states that each cell of the SSA contains exactly one symbol. The other six sets S correspond to the six conditions listed above in the definition of SSS(n). For example, $S = \{\text{fr}, \text{sc}, \text{s}, \text{t}\}$ corresponds to condition $f(i, \_, \_, j)$. Thus an SSS(n) produces an SSA(n). But the construction is reversible, so also an SSA(n) produces an SSS(n). □

We'll need some background before proceeding; see [9] for more detail. A *Latin square* of *order* n, LS(n), is an array, with rows and columns indexed by N, filled by symbols from

N, so that each row and each column contains each symbol exactly once. Thus a SS(n) is a LS(n^2). Two such squares, A and B, are said to be orthogonal, if for all $(i, j) \in N^2$, there is a unique pair $(x, y) \in N^2$ so that cell (x, y) contains symbol i in square A, and symbol j in square B. Such a pair is known as POLS(n). Here is an example for $n = 3$.

3	1	2
1	2	3
2	3	1

1	3	2
2	1	3
3	2	1

Figure 5:

Mutually orthogonal Latin squares were introduced by the great 18th century Swiss mathematician Leonhard Euler. He produced POLS(n) for all n odd, and all n a multiple of 4. He observed that there is none with $n = 2$, and conjectured the same for any n congruent to 2 modulo 4. Nearly two centuries later, Bose, Shrikhande and Parker [3], proved there is POLS(n) if and only if n is not 2 or 6.

Here is an alternate definition. An n^2 by 4 orthogonal array, which we here abbreviate to OA(n), is an array with 4 columns indexed by $C = \{\mathrm{r}, \mathrm{c}, \mathrm{s}, \mathrm{t}\}$, with the property that the n^2 by 2 subarray indexed by S is complete, for every two-element subset S of C.

Theorem 3. *There are* POLS(n) *if and only if there is an* OA(n)*.*

Proof. Here's the correspondence: if (x, y, i, j) is a row of the OA(n), fill cell (x, y) with symbol i in the first square, and symbol j in the second square. We leave the details to the reader. □

As an example, the POLS(3) in Figure 5 gives the following OA(3):

1	1	3	1
1	2	1	3
1	3	2	2
2	1	1	2
2	2	2	1
2	3	3	3
3	1	2	3
3	2	3	2
3	3	1	1

Figure 6:

Theorem 4. *If there is a* SSS(n)*, then there are* POLS(n)*.*

Proof. For an SSA(n) A, let $@ \in N$, and define an n^2 by 4 array B as follows: for each row $(w, x, y, z, @, @)$ of A, (w, x, y, z) is a row of B. It is straightforward to show B is an OA(n), which in turn yields POLS(n). □

As an example, the SSS(3) in Figure 1, gives the OA(3) in Figure 6, upon taking (@, @) to be symbol 9.

While a SSA(n) contains $6n^4$ symbols, and POLS(n) only $4n^2$ symbols, a construction in the other direction seems unlikely. And yet:

Theorem 5. *If there are* POLS(n)*, then there is a* SSS(n)*.*

Proof. We take N to be $\mathbb{Z}_n$, the group of integers modulo n. Let B be POLS(n). We construct a SSA(n) A as follows: for each row (w, x, y, z) of B, and each $a, b \in N$, take $(w, x, y + a, z + b, a, b)$ to be a row of A.

We must show that the n^4 subarray of A with columns indexed by S is complete, for each of the seven subsets S listed above in the definition of SSA(n). The seven verifications are all similar; we content ourselves with the proof for $S = \{\mathrm{fr}, \mathrm{sc}, \mathrm{s}, \mathrm{t}\}$. We need to show that if $i, j, k, l \in N$, then there is a unique row (w, x, y, z) of B, and also a unique pair $(a, b) \in N^2$, so that $(w, x, y + a, z + b, a, b) = (i, p, q, j, k, l)$, for some $p, q \in N$. So a must equal k, and b must equal l. Also, we must have $w = i$, and $z + b = j$, whence $z = jb = jl$. But B is POLS(n), so there are unique $x, y \in N$ so that $(w, x, y, z) = (i, x, y, jl)$ is a row of B. □

Note 6. Of the fifteen four-element subsets S of $\{\mathrm{fr}, \mathrm{sr}, \mathrm{fc}, \mathrm{sc}, \mathrm{s}, \mathrm{t}\}$, the corresponding subarray of A is complete for thirteen of them, not just the seven required above.

Corollary 7. *If n is a positive integer, there is a* SSS(n) *if and only if n is not* 2 *or* 6*.*

Proof. This follows from the Bose-Shrikhande-Parker result cited above. □

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