Super Sudoku Squares

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Abstract

We determine the spectrum for a variant of a completed Sudoku puzzle.

I make the bold assumption that everyone is familiar with Sudoku puzzles. Here is a completed one:

We coordinatize this as follows. The rows are divided into three groups, called fat rows, (fr), labeled 1, 2, 3. Each fat row consists of three skinny rows, (sr), also labelled 1, 2, 3. Similarly, we have fat columns, (fc), and skinny columns, (sc), also similarly labelled. Each of the 81 occupied cells is assigned a label of 4 coordinates, (fr,sr,fc,sc). For example, in
the above array, symbol 9 occurs in these 9 cells: (1, 1, 3, 1), (1, 2, 1, 3), (1, 3, 2, 2), (2, 1, 1, 2),
(2, 2, 2, 1), (2, 3, 3, 3), (3, 1, 2, 3), (3, 2, 3, 2), (3, 3, 1, 1).

To be a valid Sudoku square, each row, and each column must contain each symbol
exactly once. Furthermore, each block (= intersection of a fat row and a fat column) must
contain each symbol exactly once. Here’s the definition for other values of 3 (-):

Let $N$ be a finite non-empty set of size $|N| = n$. (We took $N = \{1, 2, 3\}$ above.) A
Sudoku square of order $n$, $SS(n)$, is a function $f : N^4 \rightarrow S$, where $S$ is a set of $n^2$ symbols,
satisfying the following properties. (Here $f(c)$ is the symbol occupying cell $c$.) Each of the
following functions is a bijection from $N^2$ to $S$: for all $i, j \in N$,

\[
(x, y) \rightarrow f(i, j, x, y),
\]
\[
(x, y) \rightarrow f(x, y, i, j),
\]
\[
(x, y) \rightarrow f(i, x, j, y).
\]

To simplify notation, we denote these three properties by $f(i, j, _, _)$, $f(_, _, i, j)$, $f(i, _, j, _)$
respectively.

We leave the proof of the following to the reader:

**Theorem 1.** There is a $SS(n)$ for every positive integer $n$.

The three properties above are defined by choosing two of the four blanks in $f(\ , \ , \ , \ )$
to be replaced by underscores _. But there are six such choices. Accordingly, we define a
Super Sudoku square of order $n$, $SSS(n)$, to be a function $f : N^4 \rightarrow S$, satisfying, for all $i, j \in N$, all six of the properties

\[
f(i, j, _, _),
\]
\[
f(_, _, i, j),
\]
\[
f(i, _, j, _),
\]
\[
f(_, _. _, j),
\]
\[
f(_, i, j, _),
\]
\[
f(_, _, i, j).
\]

Do these exist? The $SS(3)$ above is actually a $SSS(3)$! Let’s check the last three properties
at $i = 3$, $j = 2$. 

Figure 2: $f(3, _, _, 2)$

Figure 3: $f(_, 3, 2, _)$. 
Note that in all three cases, the nine occupied cells contain each of the nine symbols exactly once. Those are three of the twenty-seven properties; we leave the rest to the reader to check.

So for which positive integer \( n \) is there a \( \text{SSS}(n) \)? Try \( n = 2 \), and you’ll fail. So 2 is out.

To investigate this question, we need an alternate definition of super sudoku squares. For this, we will take \( S \) to be \( N \times N \) from now on, so our symbols are ordered pairs of elements of \( N \). For a positive integer \( k \), we say an \( n^k \) by \( k \) array is complete if its rows consist of the \( n^k \) vectors with \( k \) coordinates from \( N \), each one occurring exactly once. We define \( A \) to be an \( n^4 \) by 6 Super Sudoku array of order \( n \), \( \text{SSA}(n) \), if its six columns are indexed by the set \( \{ \text{fr, sr, fc, sc, s, t} \} \), and the \( n^4 \) by 4 subarray consisting of the columns of \( A \) indexed by \( S \) is complete, for each of the 7 following sets \( S \): \( \{ \text{fr, sr, fc} \}, \{ \text{fr, sr, s, t} \}, \{ \text{fc, sc, s, t} \}, \{ \text{fr, fc, s, t} \}, \{ \text{fr, sc, s, t} \}, \{ \text{sr, fc, s, t} \}, \{ \text{sr, sc, s, t} \} \).

**Theorem 2.** There is a \( \text{SSS}(n) \) if and only if there is a \( \text{SSA}(n) \).

**Proof.** First, suppose there is a \( \text{SSS}(n) \). We construct a \( \text{SSA}(n) \) as follows: for all \( (i, j, k, l) \in N^4 \), if cell \( (i, j, k, l) \) contains symbol \( (a, b) \), then \( (i, j, k, l, a, b) \) is a row of the \( \text{SSA} \). The condition imposed by the first set \( S \) listed above states that each cell of the \( \text{SSA} \) contains exactly one symbol. The other six sets \( S \) correspond to the six conditions listed above in the definition of \( \text{SSS}(n) \). For example, \( S = \{ \text{fr, sc, s, t} \} \) corresponds to condition \( f(i, _, _, j) \). Thus an \( \text{SSS}(n) \) produces an \( \text{SSA}(n) \). But the construction is reversible, so also an \( \text{SSA}(n) \) produces an \( \text{SSS}(n) \).
We’ll need some background before proceeding; see [2] for more detail. A *Latin square of order* \( n \), \( \text{LS}(n) \), is an array, with rows and columns indexed by \( N \), filled by symbols from \( N \), so that each row and each column contains each symbol exactly once. Thus a \( \text{SS}(n) \) is a \( \text{LS}(n^2) \). We say two such squares, \( A \) and \( B \), are said to be orthogonal, if for all \( (i, j) \in N^2 \), there is a unique pair \( (x, y) \in N^2 \) so that cell \((x, y)\) contains symbol \( i \) in square \( A \), and symbol \( j \) in square \( B \). Such a pair is known a \( \text{POLS}(n) \). Here is an example for \( n = 3 \).

\[
\begin{array}{ccc}
3 & 1 & 2 \\
1 & 2 & 3 \\
2 & 3 & 1 \\
\end{array}
\quad\quad
\begin{array}{ccc}
1 & 3 & 2 \\
2 & 1 & 3 \\
3 & 2 & 1 \\
\end{array}
\]

*Figure 5*

POLSSs were introduced by the great 18th century Swiss mathematician Leonhard Euler. He produced a \( \text{POLS}(n) \) for all \( n \) odd, and all \( n \) a multiple of 4. He observed that there is none with \( n = 2 \), and conjectured the same for any \( n \) congruent to 2 modulo 4. Nearly two centuries later, Bose, Shirkhande and Parker [1], proved there is a \( \text{POLS}(n) \) if and only if \( n \) is not 2 or 6.

Here is an alternate definition. An \( n^2 \) by 4 orthogonal array, which we here abbreviate to \( \text{OA}(n) \), is an array with 4 columns indexed by \( C = \{r, c, s, t\} \), with the property that the \( n^2 \) by 2 subarray indexed by \( S \) is complete, for every two-element subset \( S \) of \( C \).

**Theorem 3.** There is a \( \text{POLS}(n) \) if and only if there is an \( \text{OA}(n) \).

**Proof.** Here’s the correspondence: if \((x, y, i, j)\) is a row of the \( \text{OA}(n) \), fill cell \((x, y)\) with symbol \( i \) in the first square, and symbol \( j \) in the second square. We leave the details to the reader. \( \square \)

As an example, the \( \text{POLS}(3) \) in Figure 5 give the following \( \text{OA}(3) \):

\[
\begin{array}{cccc}
1 & 1 & 3 & 1 \\
1 & 2 & 1 & 3 \\
1 & 3 & 2 & 2 \\
2 & 1 & 1 & 2 \\
2 & 2 & 2 & 1 \\
2 & 3 & 3 & 3 \\
3 & 1 & 2 & 3 \\
3 & 2 & 3 & 2 \\
3 & 3 & 1 & 1 \\
\end{array}
\]

*Figure 6*
**Theorem 4.** If there is a SSS$(n)$, then there is a POLS$(n)$.

*Proof.* For an SSA$(n)$ $A$, let $@ \in N$, and define an $n^2$ by 4 array $B$ as follows: for each row $(w, x, y, z, @, @)$ of $A$, $(w, x, y, z)$ is a row of $B$. It is straightforward to show $B$ is an OA$(n)$.

As an example, the SSS(3) in Figure 1, gives the OA(3) in Figure 6, upon taking $(@, @)$ to be symbol 9.

While a SSA$(n)$ contains $6n^4$ symbols, and a POLS$(n)$ only $4n^2$ symbols, a construction in the other direction seems unlikely. And yet:

**Theorem 5.** If there is a POLS$(n)$, then there is a SSS$(n)$.

*Proof.* We take $N$ to be $\mathbb{Z}_n$, the ring of integers modulo $n$. Let $B$ be a POLS$(n)$. We construct a SSA$(n)$ $A$ as follows: for each row $(w, x, y, z)$ of $B$, and each $a, b \in N$, take $(w, x, y + a, z + b, a, b)$ to be a row of $A$.

We must show that the $n^4$ subarray of $A$ with columns indexed by $S$ is complete, for each of the seven subsets $S$ listed above in the definition of SSA$(n)$. The seven verifications are all similar; we content ourselves with the proof for $S = \{fr, sc, s, t\}$. We need to show that if $i, j, k, l \in N$, then there is a unique row $(w, x, y, z)$ of $B$, and also a unique pair $(a, b) \in N^2$, so that $(w, x, y + a, z + b, a, b) = (i, p, q, j, k, l)$, for some $p, q \in N$. So $a$ must equal $k$, and $b$ must equal $l$. Also, we must have $w = i$, and $z + b = j$, whence $z = j - b = j - l$. But $B$ is a POLS$(n)$, so there are unique $x, y \in N$ so that $(w, x, y, z) = (i, x, y, j - l)$ is a row of $B$. 

**Note 6.** Of the fifteen four-element subsets $S$ of $\{fr, sr, fc, sc, s, t\}$, the corresponding subarray of $A$ is complete for thirteen of them, not just the seven required above.

**Corollary 7.** If $n$ is a positive integer, there is a SSS$(n)$ if and only if $n$ is not 2 or 6.

*Proof.* This follows from the Bose-Shirkhande-Parker result cited above.

**References**
