Volume 4 (2023), Article 1 https://research.library.kutztown.edu/contact/

CONTACT
COMMUNICATIONS ON NUMBER THEORY AND COMBINATORIAL THEORY

A Pebbling Game on Powers of Paths

Garth Isaak*[1], Matthew Prudente†[2], and Joseph Marcinik‡[3]

[1]Department of Mathematics, Lehigh University
[2]Department of Science & Mathematics, Alvernia University
[3]Department of Physics, University of California Los Angeles

First submitted: August 12, 2022
Accepted: May 18, 2023
Published: June 3, 2023

Abstract

Two-Player Graph Pebbling is an extension of graph pebbling. Players Mover and Defender use pebbling moves, the act of removing two pebbles from one vertex and placing one pebble on an adjacent vertex, to win. If a specified vertex has a pebble on it, then Mover wins. If a specified vertex is pebble-free and there are no more valid pebbling moves, then Defender wins. The *two-player pebbling number* of a graph G, $\eta(G)$, is the minimum m such that for every arrangement of m pebbles and for any specified vertex, Mover can win. We specify the winning player for most powers of a path.

Keywords: Two-player graph pebbling, graph pebbling, game theory, paths, powers of graphs.
MSC 2020: 05C57, 91A46, 05C08.

*gi02@lehigh.edu
†matthew.prudente@alvernia.edu
‡jmarcinik@physics.ucla.edu

1 Introduction

In this manuscript, we will let G represent simple connected graphs on n vertices. We define the *power of a graph*. The *distance* from any two vertices $u, v \in V(G)$ is the length of the shortest path connecting u and v. The *diameter* of a graph G is the maximum distance over every pair of vertices in G. The results described relate to the power of a graph.

Definition 1.1. The k^{th} *power* of a graph, G^k is the graph with vertex set $V(G^k) = V(G)$ and edge set $E(G^k) = \{uv \mid d_G(u, v) \leq k\}$.

In traditional graph pebbling, as defined by Chung [2], we define graph pebbling terms.

Definition 1.2. For any graph G, a *configuration* C is an arrangement of pebbles to the vertices of G with $C(v)$ as the number of pebbles on vertex v. We denote $|C|$ to be the *size* of the configuration, i.e. the number of pebbles on the graph.

Definition 1.3. A *pebbling move* is a function between two configurations, $f : C \to C'$, in which

- there exists an edge, uv, such that $C'(u) = C(u) - 2$ and $C'(v) = C(v) + 1$, and
- $C'(x) = C(x), \forall\, x \neq u, v$.

In other words, a pebbling move removes two pebbles from one vertex and places one pebble on an adjacent vertex. A consequence of a pebbling move is that the new configuration C' and the previous configuration C have the relationship $|C'| = |C| - 1$. If a vertex has at least one pebble on it, we say the vertex is *pebbled*. If a vertex has no pebbles on it, we say it is *unpebbled* or *pebble-free*. In graph pebbling a single player uses a sequence of pebbling moves in order to place a pebble on a goal vertex, or *root*. If a configuration C yields a sequence of pebbling moves in which a pebble can be placed on the root, then we say the configuration C is r*-solvable*. We can now define the pebbling number of a graph.

Definition 1.4. For any graph G and a given choice of root r, we define $\pi(G, r)$, the rooted pebbling number, to be the smallest value m such that every configuration C, with $|C| \geq m$, is r-solvable. The *pebbling number* of a graph G is $\pi(G) = \max_{r \in V(G)} (\pi(G, r))$.

In [3], graph pebbling was extended to a two-player game with players Mover and Defender. The game evolves over rounds, with each player using pebbling moves on their turn. In each round, Mover's turn is first, with Defender following. Mover wins if they place a pebble on the root. Defender wins if there are no valid pebbling moves and the root is pebble-free. If a player makes a pebbling move from a vertex u to a vertex v, we say the player *pebbles* from u to v. In this variation, there are two rules.

1. Each player must take their turn.
2. If Mover pebbles from vertex u to vertex v, then Defender cannot pebble from v to u on their next turn.

In [4], a variation of Two-Player Pebbling was considered without rule 2, however this lead to severely restricting the graphs for which Mover could win. Also in [4], for a given graph G and choice of root r, a configuration C is *Mover-win* if Mover has a sequence of pebbling moves which place a pebble on the root, regardless of the pebbling moves made by Defender. If no sequence of pebbling moves exists for which Mover wins, then the configuration is *Defender-win.* As in classical pebbling, the two-player pebbling number is defined.

Definition 1.5. For any graph G and a given root r, the rooted two-player pebbling number, $\eta(G, r)$, is the minimum number m such that any configuration C with $|C| \geq m$, C is Mover-win. If for a graph G, a root r, and arbitrarily large m, there exists a configuration C' of size at least m, for which C' is Defender-win, then $\eta(G, r) = \infty$. The *two-player pebbling number* is $\eta(G) = \max_{r \in V}\{\eta(G, r)\}$.

We note a sufficient condition for infinite two-player pebbling number shown in [3].

Theorem 1.6 ([3])**.** *For a graph G, let S be a cut set of G. Label the components of $G - S$ as $G_0, G_1, \ldots G_k$ with $r \in G_0$. If for every $v \in S$, $|N(v) - V(G_0) - S| \geq 2$ and for every $x \in N(v) - V(G_0) - S$, $|N(x) - S| \geq 2$, then $\eta(G) = \infty$.*

If an initial configuration has two pebbles on any vertex in the neighborhood of r, then Mover can pebble to r and win. So, we say a *non-trivial configuration* on the vertices of G will have 0 or 1 pebbles on vertices in the neighborhood of r.

We note select results for $\pi(G)$ and $\eta(G)$.

Fact 1.7 ([6])**.** *For any graph G, $\eta(G) \geq \pi(G)$.*

Proposition 1.8 ([3])**.** *If $deg(r) = |V(G)| - 1$, then $\eta(G, r) = |V(G)|$.*

Corollary 1.9 ([3])**.** *For $n \geq 2$, we have $\eta(K_n) = n$.*

Theorem 1.10 ([4])**.** *For $n \geq 4$, we have $2^{n-1} \leq \eta(P_n) \leq \frac{3}{2} \cdot 2^{n-1} - n$.*

2 The Powers of Paths, P_n^k

We move on to look at Two-Player Pebbling on the k^{th} power of paths, P_n^k.

Definition 2.1. The k^{th} *power* of a graph, G^k is the graph with vertex set $V(G^k) = V(G)$ and edge set $E(G^k) = \{uv \mid d_G(u, v) \leq k\}$.

There is an upper limit when raising a graph to a power. The following fact describes the limit.

Fact 2.2. *If $diam(G) = d$, then G^d is complete.*

In light of Fact 2.2, we restrict our attention for k in G^k to be $k \leq diam(G)$. Also, we notice that P_n^1 is just a path on n vertices; the two-player pebbling number of P_n is covered in [4]. Hence, we will consider $k \in \{2, 3, \ldots, n-2\}$ when dealing with P_n^k.

First, we see the classical pebbling value for P_n^2, P_n^3, P_n^4, and P_n^k for all $k \leq n-2$.

Theorem 2.3 ([7]). *Let $0 \le r \le 1$. Then $\pi(P^2_{2k+r}) = 2^k + r$.*

Theorem 2.4 ([5]). *If $1 \le n \le 7$, then $\pi(P^3_n) = n$. For $n \ge 8$,*

$$\pi(P^3_n) = \begin{cases} 2^{\lfloor n/3 \rfloor} + 1 & \text{if } n \equiv 0 \pmod 3; \\ 2^{\lfloor n/3 \rfloor} + 2 & \text{if } n \equiv 1 \pmod 3; \\ 2^{\lfloor n/3 \rfloor + 1} & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Theorem 2.5 ([6]). *If $1 \le n \le 13$, then $\pi(P^4_n) = n$. For $n \ge 14$, we have $\pi(P^4_n) = 2^{\lceil (n-1)/4 \rceil} + r$ with $0 \le r \le 3$ and $n - 2 \equiv r \pmod 4$.*

Theorem 2.6 ([1]). *For $2 \le k \le n-2$, we have $\pi(P^k_n) = \max\{n,\, 2^d + n - 2 - k(d-1)\}$ with $d = diam(P^k_n)$.*

Path powers provide insight into the sufficient condition the graph structure which yields Defender-win configurations. Specifically, for P^k_n it is straightforward to check that there is no cut set S satisfying the sufficient conditions of Theorem 1.6. For $k \le n-4$, the set $S = \{v_2, \ldots, v_{k+2}\}$ with $G_0 = \{v_1\}$ nearly satisfies the conditions as does any set with $k+1$ consecutive vertices. The exception is that v_2 has only one neighbor outside of $S \cup \{G_0\}$, i.e., only one neighbor in $\{v_{k+3}, \ldots, v_n\}$. All other vertices in S have at least two neighbors in this set.

We will first cover the cases $\eta(P^k_n) = \infty$ when $n \ge 7$ and $2 \le k \le n-5$ in Subsection 2.1. The elementary cases in which the graph is complete or missing one edge, $\eta(P^{n-1}_n) = n$ for $n \ge 2$ and $\eta(P^{n-2}_n) = 2n-2$ for $n \ge 3$ are covered in Subsection 2.2. In Subsection 2.3, we show for $n \ge 9$, $\eta(P^{n-3}_n) \le 3n-5$ and in Subsection 2.4 we show for $n \ge 5$, $\eta(P^{n-4}_n) \le 3n+5$.

2.1 Infinite Two-Player Pebbling Number

Theorem 2.7. *Let $n \ge 7$. Then, we have $\eta(P^k_n) = \infty$ whenever $2 \le k \le n-5$.*

Proof. We show that $\eta(P^k_n, v_1) = \infty$ when $2 \le k \le n-5$. Hence $\eta(P^k_n) = \infty$ for $2 \le k \le n-5$. Consider any configuration with t pebbles distributed on $T = \{v_{k+3}, v_{k+4}, \ldots, v_n\}$ and no pebbles on $\{v_1, v_2, \ldots, v_{k+2}\}$. We will show that Defender has a winning strategy on this configuration, establishing the result.

The strategy for Defender will be to play so that at the end of its turn $C(v_1) = 0$, $C(v) \in \{0,1\}$ for $v \in S = \{v_2, \ldots, v_{k+1}\}$, $C(v_{k+2}) \in \{0,1,2\}$ and if $C(v_{k+2}) = 2$ then $C(v) = 0$ for some $v \in S$. Since $C(v) < 2$ for $v \in N(v_1) = S$ on Mover's turn, Mover can never place a pebble on the root. As described in the moves below Defender is never forced to place a pebble on the root. Thus, eventually no moves remain and Defender wins.

We will describe four cases that cover all possibilities for plays made by Mover and describe Defender's response to maintain the condition.

Case 1: If the condition still holds after Mover plays, then, as each vertex in T has at least two neighbors in T, Defender can play to a vertex in T and maintain the condition or no moves remain and Defender wins.

Case 2: If $C(v_{k+2}) \in \{1, 2\}$ and Mover places a second or third pebble on v_{k+2} then as v_{k+2} has at least 2 neighbors in T, Defender can play from v_2 to T leaving fewer than 2 pebbles on v_2. There is no change in S so the condition still holds.

Case 3: If $C(v_{k+2}) = 2$ initially and Mover places a second pebble on a vertex v_i in S then, as S is a clique, the condition implies that Defender can play from v_i to some pebble-free vertex v_j in S. After Defender moves $C(v_i) = 0$, $C(v_j) = 1$ and $C(v_{k+2})$ is either 0 or 2, so the condition still holds.

Case 4: If $C(v_{k+2}) \in \{0, 1\}$ initially and Mover places a second pebble on a vertex v_i in S, note that Mover could not have played from v_{k+2}. Thus Defender can play from v_i to v_{k+2}. After Defender moves $C(v_i) = 0$ and $C(v_{k+2}) \in \{1, 2\}$ so the condition still holds. (Mover might be able to play to T and maintain the condition with fewer pebbles on v_{k+2} but this is not necessary for the strategy to succeed.)

□

From the last paragraph of Theorem 2.7, it might seem that a simpler strategy for Defender would be to also maintain $C(v_{k+2}) < 2$ and avoid the extra condition when $C(v_{k+2}) = 2$. This is not possible as Mover could place 1 pebble on each vertex in $S \cup \{v_{k+2}\}$ and then play from v_{k+3} to v_3. In this case Defender cannot play from v_3 back to v_{k+3} so must place a second pebble on v_{k+2} to avoid placing a second pebble on $N(v_1)$ and allowing Mover to win. Notice that this provides a hint for a winning strategy for Mover on P_n^{n-4}.

Corollary 2.8. *If P_n is the path $v_1, v_2, \ldots, v_n$, then $\eta(P_n^k, v_j) = \infty$ when $2 \le k \le j - 5$ or $2 \le k \le n - j - 4$.*

2.2 P_n^{n-1} and P_n^{n-2}

We next cover two elementary cases in which Mover has a winning strategy. For $k = n - 1$, P_n^k is a complete graph so we get the following using Corollary 1.9.

Fact 2.9. *If $k \ge n - 1$, then $\eta(P_n^k) = \eta(K_n) = n$.*

Now, we move on to $k = n - 2$. Note that $P_n^{n-2} = K_n - e$ for $e = v_1 v_n$.

Theorem 2.10. *If $n \ge 3$, then $\eta(P_n^{n-2}) = 2n - 2$.*

Proof. Note that $P_n^{n-2} = K_n - e$ for $e = v_1 v_n$. If the root r is not v_1 or v_n then $\eta(P_n^{n-2}, r) = n$ by Proposition 1.8. When $n = 3$ we have $P_3^1 = P_3$. It is straightforward to check that $\eta(P_3) = 4$.

Suppose $n \geq 4$. Let $S = \{v_2, \ldots, v_{n-1}\}$.

We first show that $\eta(P_n^{n-2}, v_1) > 2n - 3$ by describing a configuration with $2n - 3$ pebbles for which Defender has a winning strategy. Place $2n - 3$ pebbles on v_n. There are initially $n-2$ possible pebbling moves from v_n. If Mover places a second pebble on a vertex in S then Defender pebbles from that vertex to an unpebbled vertex in S. Otherwise Defender pebbles from v_n to an unpebbled vertex in S. This will decrease the number of pebbling moves from v_n by at least 1. As there are $n - 2$ pebbling moves from v_n and $n - 2$ pebble-free vertices in S, there will always be a pebble-free vertex for Defender to pebble to. After $n - 2$ moves every vertex has at most 1 pebble and Defender wins.

To show $\eta(P_n^{n-2}, v_1) \leq 2n-2$ we describe a winning strategy for Mover for all configurations with at least $2n-2$ pebbles. Consider any configuration with at least $2n-2$ pebbles. If any vertex in S has at least 2 pebbles Mover wins on its first move. So assume that s vertices of S are unpebbled, $n - 2 - s$ have 1 pebble, and $C(v_n) \geq 2n - s$. There are $n - \lceil s \rceil$ pebbling moves from v_n. If Defender ever places a second pebble on a vertex of S, Mover can pebble to the root and win. Otherwise, Mover pebbles to an unpebbled vertex in S. This will decease the number of pebble-free vertices in S by at least 1 each round. As the number of moves is greater than the number of unpebbled vertices in S, either Defender will place a second pebble on a vertex of S and Mover wins as above or all vertices of S will have a pebble, Mover will place a second pebble on some vertex, moving from v_n. Since Defender cannot pebble back to v_n it must either pebble to the root or place a second pebble on a vertex in S. In each case Mover wins.

By symmetry, $\eta(P_n^{n-2}, v_1) = \eta(P_n^{n-2}, v_n) = 2n - 2$ and for $n \geq 4$, we note that $2n - 2 \geq n$. Thus the result follows. □

2.3 P_n^{n-3}

Notice that Theorem 1.10 implies that $\eta(P_4^1) = 8$. So, we may assume that $n \geq 5$.

Now, we let $S = \{v_2, v_3, \ldots, v_{n-2}\}$ and $T = \{v_{n-1}, v_n\}$. Observe that S is a clique, v_{n-1} is adjacent to every vertex of S and v_n is adjacent to every vertex of $S - \{v_2\}$.

In this section, we consider a configuration with $C(v_{n-1}) = x$ and $C(v_n) = y$. We define s to be the number of unpebbled vertices in S. We first look at nontrivial configurations based of the values of s and corresponding values for x, y. We will write (x, y) to denote starting configurations with x pebbles on v_{n-1} and y pebbles on v_n.

Lemma 2.11. *Let $n \geq 5$. Consider a configuration on P_n^{n-3} such that there are s unpebbled vertices on the vertices of $S = \{v_2, v_3, \ldots, v_{n-2}\}$ and $T = \{v_{n-1}, v_n\}$ has $C(v_{n-1}) = x$ and $C(v_n) = y$ pebbles. If $x + y \geq 3s + 4$, then Mover has a strategy to place a pebble on each vertex in S.*

Proof. If $s = 0$, we are done. So, suppose $s \geq 1$. If Defender places a second pebble on any vertex in S, then Mover will win on their next turn. So we will not consider these moves. Since v_n is not adjacent to v_2, we can divide this into two cases.

> *Case 1:* Suppose that $C(v_2) = 1$. Then Mover can pebble from either v_{n-1} or v_n to the unpebbled vertices of S. If Mover pebbles from T to S and Defender pebbles within T, then each round removes 3 pebbles from T. After s rounds, each vertex of S is pebbled and there are at least $3s+4-3s = 4$ pebbles on the vertices of T.

> *Case 2:* Suppose that $C(v_2) = 0$. Mover can pebble from v_n to any of the $s-1$ unpebbled vertices in S not v_2. After $s-1$ rounds, the only unpebbled vertex in S is v_2 and there are at least $3s + 4 - 3(s-1) = 7$ pebbles on the vertices of T. If $x \geq 2$, then Mover will pebble to v_2. If $x = 1$, Mover will pebble to v_{n-1}. Since Defender cannot pebble from v_{n-1} to v_n, they must pebble from v_n to v_{n-1}. Mover will pebble from v_{n-1} to v_2. If $x = 0$, then Mover and Defender will pebble from v_n to v_{n-1}. Mover will then pebble to v_2.

□

Theorem 2.12. *If $n \geq 8$, then $\eta(P_n^{n-3}, v_1) \leq 3n - 5$.*

Proof. Consider configurations with at least $3n-5$ pebbles distributed with s unpebbled vertices in $S = \{v_2, v_3, \ldots, v_{n-2}\}$, and t pebbles on $T = \{v_{n-1}, v_n\}$ with $C(v_{n-1}) = x$ and $C(v_n) = y$. Note that $0 \leq s \leq n-3$ and that the total number of pebbles is $t + (n-3-s)$. Assume we have a nontrivial configuration on the vertices of P_n^{n-3}. Notice that since $t + (n-3-s) \geq 3n-5$, we have $t \geq 2n + s - 2 \geq 2s + 6 + s - 2 = 3s + 4$. We note that if Defender ever places a second pebble on a vertex in S, then Mover will pebble to v_1 and win. So, we consider other pebbling options for Defender when available.

If $C(v_2) = 1$, then by Lemma 2.11, Mover can place a pebble on each vertex in S with at least 4 pebbles left on T. If $x = 2$, Mover will pebble from v_{n-1} and place a second pebble on v_2. No matter what pebbling move Defender makes, there will be a vertex in $N(v_1)$ with at least two pebbles on it. Mover will then pebble to v_1 and win.
If $C(v_2) = 0$, then by Lemma 2.11, Mover can place a pebble on each unpebbled vertex of S, other than v_2, leaving at least 7 pebbles on the vertices of T.

> *Case 1:* Let $x \in \{6, 7\}$. Then Mover pebbles to v_2 and Defender pebbles in T. Now, $x \geq 2$. Mover will pebble from v_{n-1} to v_2. No matter what pebbling move Defender makes, there will be a vertex in $N(v_1)$ with at least two pebbles on it. Mover will then pebble to v_1 and win.

> *Case 2:* Let $x \in \{4, 5\}$. Then Mover pebbles to v_2 and Defender pebbles in T. If $x \geq 2$, then Mover will pebble from v_{n-1} to v_2. No matter what pebbling move Defender makes, there will be a vertex in $N(v_1)$ with at least two pebbles on it. Mover will then pebble to v_1 and win. If $x = 0$, then $y = 6$. Mover and Defender will pebble from v_n to v_{n-1}. Now, $x = 2$. Mover will pebble from v_{n-1} to v_2. No

matter what pebbling move Defender makes, there will be a vertex in $N(v_1)$ with at least two pebbles on it. Mover will then pebble to v_1 and win.
Case 3: Let $x \in \{2,3\}$. Then, Mover pebbles to v_2 and Defender pebbles from v_n to v_{n-1}. Mover, then pebbles from v_n to v_{n-1}. Now, $x \in \{2,3\}$ again, but it is Defender's turn. Since Defender cannot pebble to v_n, they must place a second pebble on a vertex in S. Thus, Mover will pebble to v_1 and win.
Case 4: Let $x \in \{0,1\}$. Then, Mover and Defender will pebble from v_n to v_{n-1}. At this point $x \in \{2,3\}$. Mover will pebble from v_{n-1} to v_n. Since there is a pebbling move on v_n, Defender must pebble from v_n and place a second pebble on a vertex in S (they cannot pebble to v_{n-1}. Mover will pebble from S to v_1 and win.

□

Corollary 2.13. *If $n \geq 9$, then $\eta(P_n^{n-3}) \leq 3n-5$.*

Proof. If the root is v_i for $2 < i < n-1$ then $N(v_i) = V(P_n^{n-3})$ and n pebbles suffice for Mover to win since root is adjacent to all other vertices by Proposition 1.8. If root is v_2 or v_{n-1} then either v_1 has at least two pebbles and Mover wins or $3n-6$ pebbles are on the rest of the graph and Mover wins by Theorem 2.10. If $r = v_1$ or v_n, then Mover wins by Theorem 2.12. □

2.4 P_n^{n-4}

Notice that Theorem 1.10 implies that $16 \leq \eta(P_5^1) \leq 19$. So, we may assume that $n \geq 6$.

Let $S = N(v_1) = \{v_2, v_3, \ldots, v_{n-3}\}$ and $T = \{v_{n-2}, v_{n-1}, v_n\}$. Observe that S is a clique, v_{n-2} is adjacent to S, v_{n-1} is adjacent to $S - \{v_2\}$ and v_n is adjacent to $S - \{v_2, v_3\}$.

Consider a configuration with $C(v_{n-2}) = x$, $C(v_{n-1}) = y$, and $C(v_n) = z$. We will show that if $x+y+z$ is *large enough*, then Mover wins. Mover's strategy is to first place a single pebble on each vertex in S. Then Mover plays a strategy in T that forces a round such that Mover has a turn in which $C(v_{n-2}) \geq 2$. Mover then plays from v_{n-2} to v_2, placing a second pebble on v_2. As v_2 is not adjacent to v_{n-1} and v_n, Defender cannot play back to v_{n-2}. Hence, Defender either leaves a second pebble on v_2 or plays from v_2 to another vertex in S. In either case Mover can play from a vertex in S to the root and win.

We first describe a strategy playing for a subgame in T for Mover to have a turn in which $C(v_{n-2}) \geq 2$.

Lemma 2.14. *Let $n \geq 6$ and assume that $T = \{v_{n-2}, v_{n-1}, v_n\}$ with configuration $C(v_{n-2}) = x$, $C(v_{n-1}) = y$, and $C(v_n) = z$ and both Mover and Defender play in T. If $x+y+z \geq 9$, Mover has a strategy to force a round with at least 2 pebbles on v_{n-2} for Mover's turn and at least $x+y+z - \lfloor \frac{y}{2} \rfloor - 6$ pebbles remain on T.*

Proof. We will write (x,y,z) for the configuration on T and say that Mover wins if it can reach a turn with $C(v_{n-2}) \geq 2$. Observe that since T is a clique, the cases (x,y,z) and (x,z,y) are identical. So we can assume $y \leq z$.

Case 1: Let $x \geq 2$. Mover wins immediately. That is Mover wins $(2, y, z)$ for all y, z. No pebbles are used.

Case 2: Let $x = 1$. The initial configuration on T is $(1, y, z)$ with $y \leq z$. Mover will play repeatedly v_{n-1} to v_n. Since Defender cannot reverse the previous move, Defender must also play v_{n-1} to v_n or place a second pebble on v_{n-2}. In the second case Mover wins. So, if $y = 4r + s$ with $s \in \{0, 1, 2, 3\}$ play will continue to a new configuration $(1, s, z + 2r)$.

If $s \in \{2, 3\}$, then from $(1, s, z + 2r)$, Mover plays v_{n-1} to v_n. As 0 or 1 pebbles remain on v_{n-1}, and Defender cannot reverse the previous move, Defender must play from v_n to v_{n-2}, placing a second pebble on v_{n-2} and Mover wins. If $s \in \{0, 1\}$ then Mover plays v_n to v_{n-1} and Defender will also (or place a second pebble on v_{n-2}) and the situation of the previous sentence applies and Mover wins.

Observe that the only cases in which the strategies above might fail to produce a Mover win are when $(1, s, z + 2r)$ is such that $s \in \{0, 1\}$ and $z + 2r \leq 4$. In this situation either there are not 2 moves possible from v_n or after both plays play v_n to v_{n-1} after Mover plays v_{n-1} to v_n there is only one pebble on v_n so no moves remain. So the strategy could fail when $y \in \{0, 1\}$ (hence $r = 0$) and $z \leq 4$. In these cases $x + y + z \leq 6$, so Mover wins if $x + y + z \geq 7$.

When $s = 0, 1, 2, 3$, the final configurations starting from $(1, 4r + s, z)$ are $(2, 0, z + 2r - 5), (2, 1, z + 2r - 5), (2, 0, z + 2r - 1), (2, 1, z + 2r - 1)$, respectively. The number of pebbles used is $2r + 4$ when $s \in \{0, 1\}$ and $2r + 2$ when $s \in \{2, 3\}$. Hence at least $x + y + z - 2r - 4 = x + y + z - \lfloor \frac{y}{2} \rfloor - 4$ pebbles remain.

Case 3: Let $x = 0$. Mover can play v_{n-1} to v_{n-2} or v_n to v_{n-2}. In each case if Defender plays to v_{n-2} then x for the next round is 2 and Mover wins. Otherwise Defender can play v_{n-1} to v_n or vice versa assuming there are at least 2 pebbles on the other vertex. When Mover plays from v_{n-1} the next round starts at $(1, y - 1, z - 2)$ or $(1, y - 4, z + 1)$ and when Mover plays from v_n the next round starts at $(1, y - 2, z - 1)$ or $(1, y + 1, z - 4)$. In all cases if $x + y + z \geq 9$, the new configuration (x', y', z') has $x' = 1$ and $x' + y' + z' \geq 7$ and Case 2 applies. In this case, 2 additional pebbles are used before moving to the Case 2 situation and at most $\lfloor \frac{y}{2} \rfloor + 6$ pebbles are used.

□

It is interesting to note some other cases not covered by Lemma 2.14. From the proof, Mover wins all cases with $x \geq 2$; all cases $(1, y, z)$ except $y \in \{0, 1\}$ and $z \leq 4$; and all cases $(0, y, z)$ with $y + z \geq 9$. For cases $(1, 1, 2)$ and $(1, 1, 3)$ Mover can win by instead playing v_n to v_{n-1}. Using these and Mover win for $(1, 0, 5)$ one can also check that Mover wins in the sense of Lemma 2.14 whenever $x + y + z \geq 6$ with $y \leq z$ except that Defender has a winning strategy for $(1, 1, 4)$ and $(0, 0, 8)$. Building to an upper bound for $\eta(P_n^{n-4})$, we see that if S has unpebbled vertices, then Mover will want to pebble from T to S. The following Lemma yields conditions for which Mover can pebble out of T and still have two pebbles on v_{n-2}.

Lemma 2.15. *Under the same assumptions as Lemma 2.14, if $x + y + z \geq 17$ and Mover plays away from T the first time $C(v_{n-2}) \geq 2$ on Mover's turn, then Mover has a strategy to force a second round with at least 2 pebbles on v_{n-2} for Mover's turn.*

Proof. We will begin by splitting the proof into cases based on the value of x.

Case 1: Suppose $x \geq 2$. Mover plays off of v_{n-2} on the first move. No matter what Defender plays in T, at least $17-3 = 14$ pebbles remain and by Lemma 2.14 Mover reaches a second move with $C(v_{n-2}) \geq 2$.

Case 2: Suppose $x = 1$. Mover plays as in Case 2 of the proof of Lemma 2.14 reaching configurations $(2, 0, z+2r-5), (2, 1, z+2r-5), (2, 0, z+2r-1), (2, 1, z+2r-1)$ depending on s. Mover plays out of T from v_{n-2} and Defender plays in T reaching configurations $(0, 1, 2r + z - 7)$ or $(1, 0, 2r + z - 7)$ when $s = 0$; $(0, 2, 2r + z - 7)$ or $(1, 1, 2r + z - 7)$ when $s = 1$; $(0, 1, 2r + z - 3)$ or $(1, 0, 2r + z - 3)$ when $s = 2$; $(0, 2, 2r + z - 3)$ or $(1, 1, 2r + z - 3)$ when $s = 3$. Mover plays from v_n to the vertex with no pebbles or in the case each has 1 plays to v_{n-1}. Defender then must either place a second pebble on v_{n-2} or play to v_{n-1} which will now have 2 or 3 pebbles. Then, Mover plays v_{n-1} to v_n, forcing Defender to play from v_n to place a second pebble on v_n. This move is possible as long as $2r + z - 7 - 4 \geq 1$. It is straightforward to check that $2r + z \geq 12$ when $x = 1$, $y + z \geq 16$ with $y \leq z$ and $r = \lfloor \frac{y}{2} \rfloor$.

Case 3: Suppose $x = 0$. If $y \geq 2$, Mover plays v_{n-1} to v_{n-2} and then after Defender plays the configuration is $(1, y - 1, z - 2)$ or $(1, y - 4, z + 1)$. An analysis of each of these similar to Case 2 with $x = 0$ so $y + z \geq 17$ shows that Mover wins except possibly the cases (x, y, z) being one of $(0, 8, 9), (0, 8, 10), (0, 7, 10), (0, 6, 11)$.

From $(0, 8, 9)$ Mover plays v_{n-1} to v_{n-2} and Defender has two options with new configuration $(1, 4, 10)$ or $(1, 7, 7)$. We will show that Mover can win both. Assuming Defender does not place a second pebble on v_{n-2}) Mover playing from v_{n-1} to v_n forces Defender to play the same move. So from $(1, 4, 10)$ Mover forces $(1, 0, 12)$ in one round then forces $(1, 2, 8)$ in another round then Mover playing from v_{n-1} to v_n forces Defender to place a second pebble on v_{n-2}, with configuration $(2, 0, 7)$. Mover plays from v_{n-2} out of T and Defender has two choices resulting in $(0, 1, 5)$ or $(1, 0, 5)$, each of which allows Mover to win again by the comments after the Proof of Lemma 2.14 (or this is easy to check directly). In a similar manner from $(0, 8, 10)$ Mover can force $(0, 1, 6)$ or $(1, 0, 6)$ and win. From $(1, 7, 7)$ Mover forces $(1, 3, 9)$ then $(2, 1, 8)$ resulting in one of $(1, 1, 6)$ or $(0, 2, 6)$ both of which allow Mover win a second time.

From $(0, 7, 10)$ Mover forces one of $(1, 3, 11)$ or $(1, 6, 8)$. From $(1, 3, 11)$ Mover forces $(2, 1, 10)$ resulting in $(1, 1, 8)$ or $(0, 2, 8)$. In both scenarios, Mover wins a second time. From $(1, 6, 8)$ Mover forces $(1, 2, 10)$ then $(2, 0, 9)$ which results in $(0, 1, 7)$ or

$(1, 0, 7)$. In either situation, Mover wins a second time.

From $(0, 6, 11)$ Mover plays v_n to v_{n-2} forcing one of $(1, 4, 10)$ or $(1, 7, 7)$ which are the same two options from $(0, 8, 9)$ and Mover wins.

Finally we must consider $y \in \{0, 1\}$. That is configurations $(0, 0, z)$ with $z \geq 17$ and $(0, 1, z)$ with $z \geq 16$. For $(0, 0, z)$ after one round the configuration is $(1, 1, -z-4)$. Then as in case 2, with $s = 1$ after Mover plays out of T the configuration is $(0, 2, 2r+z'-7)$ or $(1, 1, 2r+z'-7)$ each of which Mover wins as long as $2r+z'-7 \geq 5$ and $z' = z - 4 \geq 13$ and $r = 0$. So Mover wins. For $(0, 1, z)$ after one round the configuration is $(1, 2, -z - 4)$. Then as in case 2, with $s = 2$ after Mover plays out of T the configuration is $(0, 1, 2r + z' - 3)$ or $(1, 0, 2r + z' - 3)$ each of which Mover wins as long as $2r + z' - 7 \geq 5$ and $z' = z - 4 \geq 12$ and $r = 0$. So Mover wins.

□

To just show finite $\eta(P_n^{n-4})$ we could obtain a second move as in Lemma 2.15 when $x+y+z \geq 25$ by applying Lemma 2.14 twice with a much shorter proof. However the final upper bound would not be as good.

Lemma 2.16. *Consider configurations on P_n^{n-4} with $n \geq 6$, s unpebbled vertices in $S = \{v_3, v_4, \ldots, v_{n-3}\}$ (when $n = 5$, $S = \emptyset$), and at least t pebbles on the remaining vertices $T = \{v_{n-2}, v_{n-1}, v_n\}$. Assuming that Defender does not place a second pebble on a vertex in S,*

(a) Mover has a strategy to place a single pebble on each vertex in S within s rounds when $s \geq 4$ and $t \geq 3s + 1$;

(b) Mover has a strategy to place a single pebble on each vertex in S within 3 rounds when $s = 1$ and $t \geq 7$; and

(c) Mover has a strategy to place a single pebble on each vertex in S within $s + 1$ rounds when $s \in \{2, 3\}$ and $t \geq 3s + 4$.

Proof. Again, we denote $C(v_{n-2}) = x$, $C(v_{n-1}) = y$, and $C(V_n) = z$. When referring to a configuration on v_{n-2}, v_{n-1}, and v_n, we will use the notation (x, y, z) Note that every vertex in S is adjacent to every vertex in T except that v_n is not adjacent to v_3.

Notice that in each part, if $C(v_3) = 1$, Mover can play from any vertex in T to any unpebbled vertex in S on each turn as long as there is a move. If Mover plays to S and Defender plays in T the number of pebbles on T is reduced by 3 in each round. So at the start of round j, at least $3s + 1 - 3(j - 1)$ pebbles remain on T. In particular there are at least 4 pebbles at the start of each round, so Mover can play as described and fill S in s rounds.

If $C(v_3) = 0$ and $x \geq 2$ or $y \geq 2$ then Mover plays first to v_3 from v_{n-2} or v_{n-1} and uses the remaining rounds to fill S as above in s rounds. If $C(v_3) = 1$ initially the same strategy works. This covers the case $s = 0$.

Proof of (a): If $s \geq 4$ and both $x, y \leq 1$, Defender must place a second pebble on one of v_{n-2} or v_{n-1} by the start of the fourth round. Then Mover plays to v_3 and completes as describe above.

Proof of (b): If $s = 1$, then we consider where the unpebbled vertex in S is. Let $C \geq 7$. If $C(v_3) = 1$, then Mover pebbles to the unpebbled vertex in S and Defender pebbles in T. This reduces C to 4. If $C(v_3) = 0$, we have 3 cases.

> *Case 1:* If $x \geq 2$ or $y \geq 2$, then Mover pebbles to v_3 and Defender pebbles in T. Now, the vertices of T have 4 pebbles on them.
>
> *Case 2:* If $x = 1$ or $y = 1$, then Mover pebbles from v_n to the unpebbled vertex. Defender will pebble then place a second pebble on v_{n-1} or v_{n-2}, else Defender pebbles to S and loses on the next round. Mover will then pebble to v_3. If $x = y = 1$, then Mover will pebble to either v_{n-1} or v_{n-2}, say v_{n-1}. If Defender pebbles from v_n to another vertex in T, then Mover will pebble to v_3. If Defender pebbles from v_{n-1} to v_{n-2}, then Mover will pebble to v_3.
>
> *Case 3:* Finally, suppose $x = y = 0$. Mover and Defender will pebble to the unpebbled vertices in T. Now, the configuration of pebbles on the vertices of T is $(1, 1, 5)$. Based on Case 2, Mover can pebble to v_3.

Proof of (c): Let $t \in \{2, 3\}$. The case when $C(v_3) = 1$ is handled above. So, suppose $C(v_3) = 0$. If both $x, y \leq 1$, Mover pebbles to the unpebbled vertices in S that are not v_3. Defender will pebble in T. This takes $s - 1$ rounds and removes $3(s - 1)$ pebbles from T. Now, the only pebble-free vertex in S is v_3 and there are at least $3s + 4 - 3(s - 1) = 7$ pebbles on the vertices of T. By the proof of part (b), Mover can place a pebble on v_3. □

From these results, we find an upper bound for $\eta(P_n^{n-4})$.

Theorem 2.17. *Let $n \geq 5$. Then $\eta(P_n^{n-4}, v_1) \leq 3n + 5$.*

Proof. Consider configurations with at least $3n+5$ pebbles distributed with s unpebbled vertices in $S = \{v_3, v_4, \ldots, v_{n-3}\}$ and t pebbles on $T = \{v_{n-2}, v_{n-1}, v_n\}$. Note that $0 \leq s \leq n-5$ and that the total number of pebbles is $t + (n - 5 - s)$ or $t + (n - 5 - s) + 1$ depending on $C(v_2)$. Assume we have a nontrivial configuration on the vertices of P_n^{n-4}.

If $C(v_2) = 1$, then Lemma 2.16 applies and we are done.

If $C(v_2) = 0$, then $t + (n - 5 - s) \geq 3n + 5$ hence $t \geq 2n + s + 10$.

If $s \geq 4$, by Lemma 2.16, after at most s rounds each vertex in S has one pebble. Each round reduces the number of pebbles on T by 3, so at least $2n + s + 10 - 3s = 2(n - s) + 10$ pebbles remain on T. Since $s \leq n - 5$ there are at least 20 pebbles remaining on T after S

is full. By Lemma 2.15, Mover wins.

If $s = 0$, then $2n + s + 10 \geq 20$ and, by Lemma 2.15, Mover wins.

If $s = 1$, then by Lemma 2.16, after at most 3 rounds, each vertex in S has one pebble. This removes at most 6 pebbles from the vertices of T, so at least $2n+s+10-6 = 2n+5$. Since $n \geq 6$, $2n + 5 \geq 17$. Lemma 2.15, Mover wins.

If $s \in \{2, 3\}$, by Lemma 2.16, after at most $s + 1$ rounds each vertex in S has one pebble. There are at least $2n + s + 10 - 3(s - 1) - 6 = 2(n - s) + 7$. Since $n - s \geq 5$, then $2(n - s) + 7 \geq 17$ and, by Lemma 2.15, Mover wins.

□

It may be possible to improve the bound slightly by a careful analysis of play as in Lemma 2.16. The conclusions of the Lemma apply for most cases of (x, y, z) with $x + y + z = 16$ and Mover might have a strategy to avoid these.

Corollary 2.18. *Let $n \geq 5$. Then $\eta(P_n^{n-4}) \leq 3n + 5$.*

Proof. If $n = 5$, then P_n^{n-4} is just a path and the inequality holds by Theorem 1.10. So, let $n \geq 6$. If $r = v_i$ for $3 < i < n - 2$, then $N[v_i] = V(P_n^{n-4})$ and n pebbles suffice for Mover to win by Proposition 1.8. If $r = v_3$, then either there is one pebbling move on the vertices v_1, v_2, in which case Mover wins, or there are at least $3n+3$ pebbles on the rest of the graph and Mover wins by $k = n - 2$ case. Notice the case of $r = v_{n-2}$ is equivalent. If $r = v_2$, then either v_1 has at least two pebbles on it and Mover wins or there are at least $3n + 4$ pebbles on the rest of the graph and Mover wins by $k = n - 3$ case. Notice the case of $r = v_{n-1}$ is equivalent. If $r = v_1$ or v_n, then Theorem 2.17 states Mover wins. □

3 Conclusion

Two-Player Pebbling still has many interesting open questions. While [4] considers the two-player pebbling number of the *join* of certain graphs, the *Cartesian product* of graphs remains open. If r is the bottom left vertex, as in Figure 1, then it is easy to see that for grids, $P_n \square P_m$, if $m, n \geq 4$, then $\eta(P_n \square P_m) = \infty$ by Theorem 1.6 using $S = N(r)$. The cases when $m, n \in \{2, 3\}$ and when $m < 4 \leq n$ remain challenging. Other, higher dimensional Cartesian products are still open, such as hypercubes Q_n.

Throughout this paper, we have used the sufficient condition for the structure of a graph for which Defender has a winning strategy in Theorem 1.6 by [3]. We have also seen graph structures in P_n^k, with $n \geq 7$ and $2 \leq k \leq n - 5$, for which $\eta(P_n^k) = \infty$. However, the proof of Theorem 2.7 does not rely on the sufficient condition stated in Theorem 1.6. We hope to improve upon the result of Theorem 1.6 to find a broader graph structure or cut set condition or even a necessary and/or sufficient condition. We have found that Mover has a winning strategy on all other values of k in P_n^k, although for $k = n - 3$ and $k = n - 4$ we only have

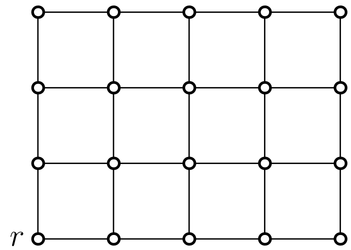

Figure 1: The Cartesian Product of $P_4 \square P_5$

upper bounds. We hope to find exact values for $\eta(P_n^{n-3})$ with $n \geq 9$ and $\eta(P_n^{n-4})$ with $n \geq 6$.

Two-player pebbling lends itself to future generalizations. One such generalization is allowing Mover to win if they place a pebble on one of any set of goal vertices R. Other generalization is allowing Defender to forfeit their turn. This could alleviate issues where Defender is forced to pebble in such a way that helps Mover.

References

[1] L. Alcón and G. Hurlbert, Pebbling in powers of paths, *Discrete Math.* **346** (2023), 113315.

[2] F.R.K. Chung, Pebbling in hypercubes, *SIAM J. Discrete Math.* **2** (1989), 467–472. http://dx.doi.org/10.1137/0402041.

[3] G. Isaak, M. Prudente, Two-Player Pebbling on Diameter 2 Graphs, *Internat. J. Game Theory* **50** (2021), 581–596. https://doi.org/10.1007/s00182-021-00766-0.

[4] G. Isaak, M. Prudente, A. Potylycki, W. Fagley, J. Marcinik, On Two-Player Pebbling, *Commun. Number Theory Combin. Theory* **3** (2022), Article 3.

[5] J. Kim, Pebbling Exponents of Graphs, *J. of Natural Sciences In Catholic Univ. of Daegu* **2** (2004), 1–7.

[6] J. Kim, S. Kim, Pebbling Exponents of Paths, *Honam Math. J.* **32** (2010), 769–776.

[7] L. Pachter, H. Snevily, B. Voxman, On pebbling graphs, *Congr. Numer.* **107** (1995), 65–80.