Volume 3 (2022), Article 5 https://research.library.kutztown.edu/contact/

CONTACT
COMMUNICATIONS ON NUMBER THEORY AND COMBINATORIAL THEORY

Meertens number and its variations

Chai Wah Wu*[1]

[1]IBM Research, IBM T. J. Watson Research Center

First submitted: August 18, 2022
Accepted: November 29, 2022
Published: December 5, 2022

Abstract

In 1998, Bird introduced Meertens numbers as numbers that are invariant under a map similar to the Gödel encoding. In base 10, the only known Meertens number is 81312000. We look at some properties of Meertens numbers and consider variations of this concept. In particular, we consider variations of Meertens numbers where there is a finite time algorithm to decide whether such numbers exist, exhibit infinite families of these variations and provide bounds on parameters needed for their existence.

Keywords: Meertens numbers, Gödel encoding
MSC 2020: 11A63, 00A08

1 Introduction

Kurt Gödel in his celebrated work on mathematical logic [2] uses an injective map from the set of finite sequences of symbols to the set of natural numbers in order to describe statements in logic as natural numbers and relate properties of mathematical proofs with properties of natural numbers. This approach is subsequently used by Alan Turing to define

*cwwu@us.ibm.com

the notion of computable numbers [6], which are numbers that can be computed by his abstract computing model. This seminal work ushered in the field of theoretical computer science. The basic Gödel encoding is as follows: each symbol in an alphabet is mapped to a distinct positive integer. Thus a finite sequence of symbols $s_1, \ldots, s_n$ is mapped to a sequence of positive integers $m_1, \ldots, m_n$. This sequence is then mapped to the natural number $G = \prod_{i=1}^{n} p_i^{m_i}$, where p_i denotes the i-th prime.

In [1], Richard Bird dedicated the number 81312000 to his friend Lambert Meertens on the occasion of his 25 years at the Centrum Wiskunde & Informatica (CWI) institute and called it a *Meertens number*. He constructed this number using a mapping similar to the Gödel encoding.

Definition 1.1. Given a decimal representation $d_1, \ldots, d_n$ of the number $m = \sum_{i=1}^{n} d_i 10^{n-i}$, if $m = \prod_{i=1}^{n} p_i^{d_i}$ then m is called a *Meertens number*.

The only Meertens number known to date is 81312000 whose prime factorization is $2^8 3^1 5^3 7^1 11^2 13^0 17^0 19^0$ and Bird believed that there are no other Meertens number [1]. David Applegate has conducted the search up to 10^{29} [3] without finding any other Meertens number.

Definition 1.2. Let $b \geq 2$ and $0 \leq d_i < b$ with $d_1 > 0$ be integers such that $m = \sum_{i=1}^{n} d_i b^{n-i}$, then $M_b(m)$ is defined as $M_b(m) = \prod_{i=1}^{n} p_i^{d_i}$.

Thus a Meertens number is a fixed point of the function $M_{10}(\cdot)$. Note that the function M_{10} is similar to the Gödel encoding function. However, unlike the Gödel encoding, this function is not injective as d_i can be 0 for $i > 1$. In particular, $M_{10}(10^k) = 2$ for all $k \geq 0$. Since $d_i \leq 9$, the exponents of the primes 2 and 5 must be less than or equal to 9 and thus a Meertens number has at most 9 trailing zeros. In particular, the number of trailing zeros is the minimum of the first and third digits of m.

2 Meertens number in other bases

As noted in [1], the concept of a Meertens number can be defined in other number bases as well, i.e. m is a Meertens number in base b if m satisfies $m = \prod_{i=1}^{n} p_i^{d_i} = \sum_{i=1}^{n} d_i b^{n-i}$ for some nonnegative integers $d_i < b$ with $d_1 > 0$, i.e., $M_b(m) = m$. Since $d_1 \neq 0$, it is clear that a Meertens number must necessarily be even. Similarly, the number of trailing zeros in base 10 is the minimum of the first and third digits of m in base b. Table 1 lists some Meertens numbers found in various number bases.

The number 82944 is interesting and has a very curious property. Note that 82944 is a Meertens number in base 8294 and shares the first 4 digits with the base. The number 82944 in base 8294 is $A4$ (where we borrow from hexadecimal notation and use A to denote the digit 10) and $2^{10}3^4 = 82944$. Are there other numbers with this property? The answer is yes.

Theorem 2.1. *If* $1024 \cdot 3^c - c$ *is divisible by* 10 *for some integer* $c \geq 0$*, then* $1024 \cdot 3^c$ *is a Meertens number in base* $b = \frac{1024 \cdot 3^c - c}{10}$.

Number base	Meertens number	Number base	Meertens number
2	2, 6, 10	481	486
3	10	512	4294967296
4	200	1452	1458
5	6, 49000, 181500	1455	2916
6	54	1942	5832
7	100	4096	65536
8	216	4367	4374
9	4199040	7775	46656
10	81312000	8294	82944
14	47250	13114	13122
16	18	13118	26244
17	36	26242	104976
19	96	39357	39366
32	256	52485	157464
51	54	74649	746496
64	65536	118088	118098
71	216	209951	1679616
158	162	354283	354294
160	324	1062870	1062882
323	1296	1119743	10077696

Table 1: Meertens numbers in various number bases.

Proof. First note that $b > 10$, $b > c$ and $1024 \cdot 3^c = 10b + c$ written in base b has digits 10 and c which maps to $1024 \cdot 3^c$ under the map M_b. □

There are two solutions with $c < 10$, i.e, $c = 4$ and $c = 6$, with $c = 4$ corresponding to the number 82944 above and $c = 6$ corresponding to a Meertens number 746496 in base 74649. Similarly, $2^{100}3^{96} - 96$ is a Meertens number in base $\frac{2^{100}3^{96}-96}{100}$ and the base in decimal is equal to the Meertens number in decimal without the last 2 digits.

Since M_b is not injective, it is possible for a number to be a Meertens number in more than one base. We note in Table 1 that 6, 10, 216 and 65536 are Meertens numbers in more than one base. Are there any others? The answer is yes as a consequence of the following result.

Theorem 2.2. *If a, k and m are positive integers such that $a + km = 2^a$ and $a < k$, then 2^{2^a} is a Meertens number in base 2^k. In particular for $a > 2$, 2^{2^a} is a Meertens number in base $2^{2^a - a}$.*

Proof. Since $2^{2^a} = 2^a 2^{km}$, this means that 2^{2^a} consists of a single digit of value $2^a < 2^k$ followed by m zeros. Thus $M_{2^k}\left(2^{2^a}\right) = 2^{2^a}$. For $a > 2$, $2^a - a > a$ and by setting $m = 1$, this shows that 2^{2^a} is a Meertens number in base $2^{2^a - a}$. □

In particular, we have the following Corollary:

Corollary 2.3. *If $k > a$ is a divisor of $2^a - a$, then 2^{2^a} is a Meertens number in base 2^k.*

For small values of a we list these divisors in Table 2. This shows that there are many numbers (for example $4294967296 = 2^{2^5}$) that are Meertens numbers in more than one base. For instance $2^{2^{16}}$ is a Meertens number in at least 105 different bases and $2^{2^{64}}$ is a Meertens number in at least 435 bases! In particular, for any integer $t > 2$, $2^{2^t - k} - 2^{t-k}$ is a divisor of $2^{2^t} - 2^t$ for $k = 0, \ldots, t$. Thus $2^{2^{2^t}}$ is a Meertens number in at least $t + 1$ different bases, i.e. there are numbers which are Meertens numbers for an arbitrarily large number of bases. Even though there is only one known Meertens number in base 10, the above argument also implies that there are arbitrarily large bases for which Meertens numbers exist. What about numbers that are not powers of 2?

Theorem 2.4. *For integers $m \geq n \geq 0$,*

- *$2 \cdot 3^n$ is a Meertens number in base $2 \cdot 3^n - n$,*
- *$2^{2^n}3^{2^m}$ is a Meertens number in base $2^{(2^n - n)}3^{2^m} - 2^{m-n}$, and*
- *$2^{3^n}3^{3^m}$ is a Meertens number in base $2^{3^n}3^{(3^m - n)} - 3^{m-n}$.*

Proof. Since $2n < 2 \cdot 3^n$, $2 \cdot 3^n$ is written as $1n$ in base $b = 2 \cdot 3^n - n$, and $M_b(2 \cdot 3^n) = 2 \cdot 3^n$. Similarly, $2^{n+1} \leq 2^{m+1} < 2^{(2^n - n)}3^{2^m}$ and the 2 digits in the base $2^{(2^n-n)}3^{2^m} - 2^{m-n}$ representation of $2^{2^n}3^{2^m}$ are 2^n and 2^m which is mapped by M_b into $2^{2^n}3^{2^m}$. Next, $3^{n+1} \leq 3^{m+1} < 2^{3^n}3^{(3^m-n)}$ and the 2 digits in the base $2^{3^n}3^{(3^m-n)} - 3^{m-n}$ representation of $2^{3^n}3^{3^m}$ are 3^n and 3^m which is mapped by M_b into $2^{3^n}3^{3^m}$. □

For instance, 157464 in Table 1 is of the form $157464 = 2^{3^1}3^{3^2}$ and thus is a Meertens number in base $2^{3^1}3^{3^2-1} - 3^{2-1} = 52485$.

a	k: divisors of $2^a - a$ larger than a
3	5
4	6, 12
5	9, 27
6	29, 58
7	11, 121
8	31, 62, 124, 248
9	503
10	13, 26, 39, 78, 169, 338, 507, 1014
11	21, 97, 291, 679, 2037
12	1021, 2042, 4084
13	8179
14	1637, 3274, 8185, 16370
15	4679, 32753
16	18, 20, 21, 24, 26, 28, 30, 35, 36, 39, 40, 42, 45
17	8737, 26211, 43685, 131055
18	131063, 262126
19	524269
20	262139, 524278, 1048556
21	2097131
22	349, 698, 1047, 2003, 2094, 4006, 6009, 12018
23	45, 131, 393, 655, 1179, 1423, 1965, 4269, 5895
24	773, 1546, 2713, 3092, 5426, 6184, 10852, 21704
25	271, 123817, 33554407
26	197, 394, 170327, 340654, 33554419, 67108838
27	457, 509, 577, 232613, 263689, 293693, 134217701
28	1493, 2986, 4479, 5972, 8958, 14983, 17916, 29966
29	317, 951, 564533, 1693599, 178956961, 536870883
30	34, 38, 323, 361, 646, 722, 6137, 12274, 87481

Table 2: Values of a and k such that 2^{2^a} is a Meertens number in base 2^k. The list is not exhaustive for $a = 16, 22, 23, 24, 28, 30$; for instance $a = 16$ has 105 such divisors.

3 Injective Gödel-like encodings

As mentioned earlier, the encoding defined by $M_b(m)$ is not a proper Gödel encoding as it is not one-to-one. Next we look at some injective Gödel-like encodings.

3.1 α-Meertens number

Definition 3.1. Let $b \geq 2$ and $0 \leq d_i < b$ with $d_1 > 0$ be integers such that $m = \sum_{i=1}^{n} d_i b^{n-i}$, then $N_b(m) = \prod_{i=1}^{n} p_i^{d_i+1}$

Note that by the unique factorization theorem of the integers, N_b is one-to-one on the set of positive integers. We will call numbers such that $N_b(m) = m$ an *α-Meertens number (in base b)*. Since the encoding is one-to-one, there cannot be a number n that is a fixed point of this encoding in more than one base. This is easily seen as a number will have different digits in different bases. Some examples of α-Meertens numbers in various bases are listed in Table 3.

The following result shows that there are an infinite number of α-Meertens numbers.

Theorem 3.2. *For $t \geq 0$, $3 \cdot 2^{2^t+1}$ is an α-Meertens number in base $b = 3 \cdot 2^{2^t-t+1}$.*

Proof. First note that $2^t < 3 \cdot 2^{2^t-t+1}$. Then $3 \cdot 2^{2^t+1}$ in base $3 \cdot 2^{2^t-t+1}$ is the digit 2^t followed by the digit 0 which maps to $3 \cdot 2^{2^t+1}$ under the mapping N_b. □

On the other hand, we will next show that for a fixed b there are only a finite number of α-Meertens numbers in base b.

Definition 3.3. Let p_i denote the i-th prime number and let $p_n\#$ denote the primorial defined as $p_n\# = \prod_{i=1}^{n} p_i$. Let $\vartheta(t)$ denote the first Chebyshev function defined as $\vartheta(n) = \sum_{p \leq n} \log(p)$ where p ranges over all prime numbers less than or equal to n.

Lemma 3.4. *$p_n\# > n^{0.5972n}$. If $n \geq 947$, then $p_n\# > n^{0.980n}$.*

Proof. For $n = 1$, the statement is trivially true. For $n > 1$, note that $p_n\# = e^{\vartheta(p_n)}$. Rosser [4] showed that for $n \geq 1$, $p_n > n \log n$. In [5, Theorem 10], it was shown that for $n \geq 7481$, $\vartheta(n) > 0.980n$. For primes $2 < p_n < 7481$, a simple computation shows that $\vartheta(p_n) > 0.5972 p_n$. This implies that $p_n\# > e^{0.5972 p_n} > e^{0.5972 n \log n} = n^{0.5972n}$ for $n > 1$. The second part follows from the fact that the 947^{th} prime is 7481. □

Lemma 3.5. *If m is an α-Meertens number in base b with k digits, then $b^k > 2p_k\#$.*

Proof. Since m expressed in base b has k digits, $m < b^k$. On the other hand, by the definition of α-Meertens numbers, $m = N_b(m) \geq 2p_k\#$. □

Theorem 3.6. *If m is an α-Meertens number in base b, then $m < b^{b^{1.675}}$.*

Proof. Suppose that m expressed as a base b number has k digits. Then by Lemma 3.5 and Lemma 3.4, $b^k > 2p_k\# > k^{0.5972k}$, implying that $k < b^{1.675}$. Thus $m < b^k < b^{b^{1.675}}$. □

Corollary 3.7. *For a fixed b, let k^* be the largest integer k such that $b^k > 2p_k\#$. Then $k^* \leq b^{1.675}$. If m is an α-Meertens number in base b, then $m < b^{k^*}$.*

base	α-Meertens number
12	12, 24
16	48
24	96
35	36
64	384
106	108
107	216
115	576
192	1536
321	324
329	2304
431	1296
968	972
970	1944
1943	7776
2048	24576
2911	2916
8742	8748
8745	17496
11662	34992
24576	393216
26237	26244
46655	279936
78724	78732
78728	157464
157462	629856
236187	236196
314925	944784

Table 3: α-Meertens numbers in various bases.

Proof. This is a consequence of Lemma 3.5 and Theorem 3.6. □

Corollary 3.8. *For $b \leq 10000$, if m is an α-Meertens number in base b, then $m < b^{b-1}$. If in addition $608 \leq b$, then $m < b^{\frac{b}{2}}$.*

Proof. The computer-assisted proof requires computing the value of k^* in Corollary 3.7 for various b. □

This allows us to improve Theorem 3.6.

Theorem 3.9. *If m is an α-Meertens number in base b, then $m < b^{b^{1.02041}}$.*

Proof. Suppose m has k digits in base b. Then $m < b^k$. If $k \geq 947$, then the proof of Theorem 3.6 combined with the second part of Lemma 3.4 shows that $k < b^{1.02041}$. Suppose $k < 947$. If $b \geq 826$, then $b^{1.02041} \geq 947$ and thus again $k < b^{1.02041}$. For $b < 826$, Corollary 3.8 shows that $m < b^{b-1} < b^{b^{1.02041}}$. □

Theorem 3.10. *If m is an α-Meertens number in base b, then $m < b^{b^{1+\epsilon}}$ where $\epsilon \to 0$ as $b \to \infty$.*

Proof. It is well known that $\vartheta(x)$ behaves asympotically as x. In particular, [5, Theorem 4] shows that $\vartheta(x) > (1-\delta)x$ where $\delta \to 0$ as $x \to \infty$. The rest of the proof is similar to the proof of Theorem 3.9 to show that $k < b^{\frac{1}{1-\delta}}$. □

We conjecture that k^* grows slower than the upper bound $b^{1.02041}$ or the asymptotic upper bound $b^{1+\epsilon}$ and that Corollary 3.8 is true for all b.

Conjecture 3.11. *All α-Meertens numbers m in base b satisfies $m < b^{b-1}$ and satisfies $m < b^{\frac{b}{2}}$ for large enough b.*

The first few values of k^* as a function of b is shown in Table 4 and a plot of k^* versus b is shown in Fig. 1 where k^* appears to be less than b for all b and less than $\frac{b}{3}$ for large b.

b	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
k^*	0	0	3	4	5	6	7	8	9	10	11	12	13	14	14

Table 4: Values of k^* as defined in Corollary 3.7 for various b.

The following result shows that 12 is the smallest base for which there exists an α-Meertens number.

Theorem 3.12. *There are no α-Meertens numbers in base $b < 12$.*

Proof. This again requires a computer-assisted proof. If m is an α-Meertens number in base b, then Corollary 3.8 implies that $m < b^{b-1}$. Next an exhausive search up to b^{b-1} for $b < 12$ shows that there are no α-Meertens numbers in base $b < 12$. □

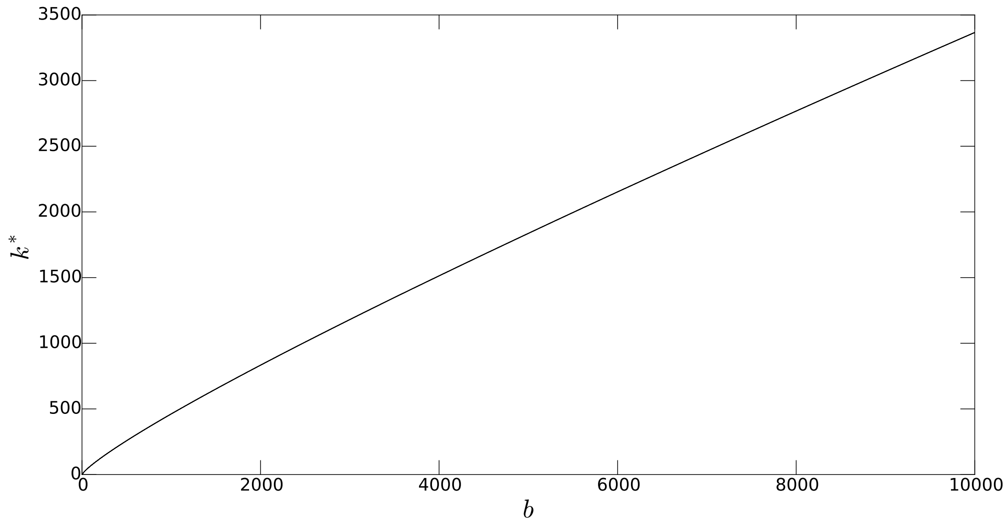

Figure 1: Plot of k^* as defined in Corollary 3.7 as a function of b.

3.2 Reverse Meertens number

Another way to define a one-to-one encoding is by reversing the digits and applying M_b, i.e. if the base-b representation of a number m is $d_n, \ldots, d_1$ with $d_n > 0$, then the encoding $M_b^r(m) = \prod_{i=1}^n p_i^{d_i}$ is one-to-one[1] and we define a *reverse Meertens number* in base b as a number m such that $M_b^r(m) = m$. As before, because this encoding is one-to-one, a number can be a reverse Meertens number in at most one base. In base 10, $12 = 3^1 2^2$ is a reverse Meertens number. Reverse Meertens numbers in different bases are listed in Table 5.

Note that 17496 is both a reverse Meertens number and an α-Meertens number (in different bases). Clearly, Meertens number such as 6, 100, 36 and 1296 which are palindromes in their respective bases (5, 7, 17 and 323) are also reverse Meertens numbers.

Theorem 3.13. *For integers $m \geq n \geq 0$,*

- $3 \cdot 2^n$ *is a reverse Meertens number in base* $3 \cdot 2^n - n$*,*
- $2^{2^m} 3^{2^n}$ *is a reverse Meertens number in base* $2^{(2^m - n)} 3^{2^n} - 2^{m-n}$ *and*
- $2^{3^m} 3^{3^n}$ *is a reverse Meertens number in base* $2^{3^m} 3^{(3^n - n)} - 3^{m-n}$*.*

Proof. Since $2n < 3 \cdot 2^n$, $3 \cdot 2^n$ is written as $1n$ in base $b = 3 \cdot 2^n - n$, and $M_b^r(3 \cdot 2^n) = 3 \cdot 2^n$. Similarly, $2^{n+1} \leq 2^{m+1} < 2^{(2^m - n)} 3^{2^n}$ and the 2 digits in the base $2^{(2^m-n)} 3^{2^n} - 2^{m-n}$ representation of $2^{2^m} 3^{2^n}$ are 2^n and 2^m which is mapped by M_b^r into $2^{2^m} 3^{2^n}$. Next, $3^{n+1} \leq 3^{m+1} < 3^{(3^n - n)} 2^{3^m}$ and the 2 digits in the base $2^{3^m} 3^{(3^n - n)} - 3^{m-n}$ representation of $2^{3^m} 3^{3^n}$ are 3^n and 3^m which is mapped by M_b^r into $2^{3^m} 3^{3^n}$. □

Theorem 3.14. $p_{r+1}^{p_{r+1}}$ *is a reverse Meertens number in base* $b = p_{r+1}^{\frac{p_{r+1}-1}{r}}$ *if r divides $p_{r+1} - 1$.*

[1] On the other hand, note that in contrast to the definition of M_b, if we remove the requirement that $d_n > 0$, and add leading zeros to the base-b representation of m, this will not affect the value of $M_b^r(m)$.

base	reverse Meertens numbers	base	reverse Meertens numbers
3	3, 10, 273	1148	2304
5	6, 175	1527	1536
7	100	2187	19683
9	27	2499	17496
10	12	3062	3072
17	36	4603	9216
21	24	4605	13824
25	3125	5182	20736
44	48	6133	6144
49	823543	7775	46656
70	144	9997	69984
71	216	12276	12288
91	96	12440	62208
97	486	18426	36864
186	192	24563	24576
194	972	36860	110592
285	576	49138	49152
323	1296	73721	147456
377	384	98289	98304
574	1728	209951	1679616
760	768	1119743	10077696

Table 5: Reverse Meertens numbers in various bases.

Proof. Since $k < k^i$ for $k, i > 1$, consider a base k^i representation consisting of the digit k followed by r zeros, where $r = \frac{k-1}{i}$. This represents the number $m = k(k^i)^r = k^{ir+1} = k^k$. Under the mapping M_b^r, $M_b^r(m) = p_{r+1}^k$. Then the result follows if $k = p_{r+1}$. □

In particular, the first few primes p_{r+1} satisfying the condition in Theorem 3.14 are: 3, 5, 7, 31, 97, 101, 331, 1009, 1093, 1117, 1123, 1129, 3067, 64621, and 480853.

4 Zeroless Meertens numbers

Next we study Meertens numbers in base b without a zero digit when written in base b representation. We will call these numbers *zeroless Meertens numbers*. Examples include 6, 18, 36, 96, 54, 216, 1296 with corresponding bases 5, 16, 17, 19, 51, 71, 323. Similarly, examples of zeroless reverse Meertens numbers are: 6, 12, 36, 24, 48, 144, 1296 with corresponding bases 5, 10, 17, 21, 44, 70, 323. In fact, Theorems 2.4 and 3.13 show that there are an infinite number of bases with zeroless Meertens numbers or with zeroless reverse Meertens numbers. On the other hand, for a fixed b, the number of zeroless Meertens numbers and zeroless reverse Meertens numbers is finite.

Theorem 4.1. *If b is squarefree, then a zeroless Meertens number or a zeroless reverse Meertens number m in base b satisfies $m < b^{u-1}$, where p_u is the largest prime dividing b.*

Proof. Suppose m is a zeroless Meertens number in base b. Let S be the set of indices of primes which divide b, i.e. $b = \prod_{i \in S} p_i$. If m has u or more digits, then $d_i > 0$ for each $i \in S$, i.e. b divides $m = \prod_i p_i^{d_i}$, and m has a trailing zero digit in base b leading to a contradiction. The case of a zeroless reverse Meertens number is similar. □

The analysis in Section 3.1 can also be used to bound the number of zeroless (reverse) Meertens numbers.

Lemma 4.2. *For a fixed b, let l^* be the largest integer l such that $b^l > p_l\#$. Then $k^* \leq l^* \leq b^{1.675}$.*

Proof. The proof is similar to the proof of Corollary 3.7. □

Theorem 4.3. *If m is a zeroless Meertens number or a zeroless reverse Meertens number in base b, then $m < b^{l^*} \leq b^{b^{1.675}}$.*

Proof. The proof is similar to the proof of Theorem 3.6. □

b	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
l^*	0	2	4	5	6	7	8	9	10	11	12	12	13	14	15

Table 6: Values of l^* for various b.

Theorems 4.1 and 4.3 and an exhaustive computer search show the following.

Theorem 4.4.
- *The number* 6 *(associated with base* 5*) is the only zeroless Meertens number for bases* $b < 12$.
- *The numbers* 6 *(associated with base* 5*) and* 12 *(associated with base* 10*) are the only zeroless reverse Meertens number for bases* $b < 12$.
- *There are no zeroless Meertens numbers or zeroless reverse Meertens numbers in bases* 13, 14, *or* 15.
- *The number* 36 *is the only zeroless Meertens number and zeroless reverse Meertens number in base* 17.

On the other hand, the number of zero digits in large Meertens numbers are large. We can estimate the number of zero digits in a Meertens or reverse Meertens number:

Theorem 4.5. *If m is a Meertens or a reverse Meertens number in base b with u digits, then the number of zero digits in m is larger than*

$$u - e^{W(1.675u\log(b))} \tag{1}$$

where W is the Lambert W function.

Proof. Let z be the number of zero digits in m. Then

$$\begin{aligned} b^u &> m \geq p_{u-z}\# > (u-z)^{0.5972(u-z)} \\ u\log b &> 0.5972(u-z)\log(u-z) \\ 1.675u\log b &> (u-z)\log(u-z) \\ u-z &< e^{W(1.675u\log b)} \end{aligned}$$

□

Since $W(x) = \log(x) - \log(\log(x)) + o(1)$, this means that for a Meertens number m, as $m \to \infty$, $\frac{e^{W(1.675u\log b)}}{u} \to 0$ and the fraction of zero digits in a Meertens number approaches 1.

5 Generalized Meertens numbers and generalized reverse Meertens numbers

Definition 5.1. Given a pair of maps $f = \{f_1, f_2\}$ where $f_1 : \mathbb{N} \to \mathbb{N}$, $f_2 : \mathbb{N} \to \mathbb{N}$, define the map

$$M^f(d_1, \ldots, d_n) = \prod_{i=1} f_1(i)^{f_2(d_i)}.$$

A generalized Meertens number (GMN) in base b is a number m such that $M^f(d_1, \ldots, d_n) = m$ where $(d_1, \ldots, d_n)$ are the digits of m in base b. A generalized reverse Meertens number (GRMN) in base b is a number m such that $M^f(d_n, \ldots, d_1) = m$.

In the cases we discussed in the sections above, $f_1(i)$ is the i-th prime and $f_2(i) = i$ or $f_2(i) = i + 1$. For these cases, since $p^d > d$ for all primes p and integers d, all GMN and GRMN in base b must be larger than or equal to b. The tables above show that it is possible for a GMN or GRMN in base b to be equal to b. In particular, 2 is a Meertens number in base 2, 12 is an α-Meertens number in base 12 and 3 is a reverse Meertens number in base 3. In fact, since b written in base b is 10, applying the digits $(1, 0)$ (resp. the digits $(0, 1)$) to M^f will return a number b which is a GMN (resp. GRMN) in base b. This is summarized in the following result.

Theorem 5.2. *Suppose $f_1(i) > i$ for all i.*

- *If c is a GMN or a GRMN in base b, then $c \geq b$.*
- *If $f_1(1)^{f_2(1)} f_1(2)^{f_2(0)} > 1$, then b is a GMN in base b where the base is given by $b = f_1(1)^{f_2(1)} f_1(2)^{f_2(0)}$.*
- *If $f_1(2)^{f_2(1)} f_1(1)^{f_2(0)} > 1$, then b is a GRMN in base b where $b = f_1(2)^{f_2(1)} f_1(1)^{f_2(0)}$.*

5.1 The case when $f_1 = f_2 = \mathrm{Id}$

Consider the case where f_1 and f_2 are both the identity maps, i.e. $f_1(i) = f_2(i) = i$. Clearly 1 is a GMN and a GRMN in this case. The encoding M^f is not injective since $1^{d_1} = 1$ for all d_1. In base 10, $324 = 1^3 2^2 3^4$ is a GMN and $64 = 2^6 1^4$ is a GRMN. Tables 7-8 list some GMN and GRMN numbers under these f_i's.

Theorem 5.3. *Suppose $f_1(i) = f_2(i) = i$. Then 2^a is a generalized Meertens number in base k if $k > a$ is a divisor of $2^a - a$ such that $k^2 > 2^a - a$.*

Proof. First note that $2^a - a > 0$ for all a. By hypothesis, there exists $0 < d < k$ such that $dk + a = 2^a$. Since $k > a$, this implies that 2^a written in base k consists of the digits d and a and is a generalized Meertens number since $M^f(d, a) = 1^d 2^a = 2^a$. □

Note that the bases k that satisfy the condition in Theorem 5.3 (Table 9) are a subset of the divisors in Table 2.

Theorem 5.4. *Suppose $f_1(i) = f_2(i) = i$. Then 2^a is a generalized reverse Meertens number in base k for all integers k in the range* $\max(a, \frac{2^a}{a+1}) < k \leq \frac{2^a}{a}$.

Proof. By hypothesis, $0 \leq 2^a - ka < k$ and $k > a$. Thus 2^a written in base k consists of the digits a and $d = 2^a - ka$ where $0 \leq d < k$ and is a generalized reverse Meertens number since $M^f(d, a) = 1^d 2^a = 2^a$. □

A consequence of Theorem 5.4 is that all powers of 2 except for $4, 8$ and 16 are GRMNs when $f_1(i) = f_2(i) = i$. Since for a fixed b, there are only a finite number of integers l such that $b^l > l!$,[2] similar arguments as in Section 4 can be used to show that the number of zeroless generalized (reverse) Meertens numbers for a fixed base b is finite for this choice of f_i's.

[2] As a consequence of Stirling's approximation, the largest such l is asymptotially on the order of eb, where e is Euler's number.

base	generalized Meertens numbers
2	1350, 47520, 1995840
3	25920
4	108, 518400, 864000
5	8, 17280
6	16, 36864
7	72
8	746496
9	32
10	324
12	16, 1458
23	1728
27	32
29	64
31	256
34	1296
39	1024
58	64
62	256
71	20736, 746496
78	1024
97	2048

Table 7: Generalized Meertens numbers in various bases for the case when f_1 and f_2 are identity maps. The number 1 is omitted as it is always a GMN for these f_i's.

base	generalized reverse Meertens numbers
2	2, 6, 12, 576000
3	120, 360
4	54, 115200
5	48, 188160
6	32
7	768
8	216, 1728
10	64
11	192, 729, 1536
16	5184
17	128
18	128, 6912
23	1728
24	768
29	256
30	256
31	256
32	256, 2304
40	8192000
50	10368
52	512
53	512
54	512
55	512
56	512
71	20736, 746496
73	6144
94	1024
95	1024
96	1024
97	1024

Table 8: Generalized reverse Meertens numbers in various bases for the case when f_1 and f_2 are identity maps. The number 1 is omitted as it is always a GRMN for these f_i's.

a	k: divisors of $2^a - a$ with $k > a$ and $k^2 > 2^a - a$
3	5
4	6, 12
5	9, 27
6	29, 58
7	121
8	31, 62, 124, 248
9	503
10	39, 78, 169, 338, 507, 1014
11	97, 291, 679, 2037
12	1021, 2042, 4084
13	8179
14	1637, 3274, 8185, 16370
15	4679, 32753
16	260, 273, 280, 312, 315, 336, 360, 364, 390, 420, 455, 468
17	8737, 26211, 43685, 131055
18	131063, 262126
19	524269
20	262139, 524278, 1048556
21	2097131
22	2094, 4006, 6009, 12018, 699047, 1398094, 2097141, 4194282
23	4269, 5895, 7115, 12807, 21345, 64035, 186413, 559239
24	5426, 6184, 10852, 21704, 2097149, 4194298, 8388596, 16777192
25	123817, 33554407
26	170327, 340654, 33554419, 67108838
27	232613, 263689, 293693, 134217701
28	17916, 29966, 44949, 59932, 89898, 179796, 22369619
29	564533, 1693599, 178956961, 536870883
30	87481, 174962, 1487177, 1662139, 2974354, 3324278

Table 9: Values of a and k such that 2^a is a generalized Meertens number in base k with $f_1 = f_2$ equal to the identity function. The list is not exhaustive for $a = 16, 23, 28, 30$; for instance $a = 16$ has 60 such divisors.

6 Conclusions

We study Meertens numbers and their variations which are defined as fixed points of maps on the natural numbers. Depending on the map, the set of such numbers can be sparse or abundant. We showed that for α-Meertens numbers and zeroless (reverse) Meertens numbers these numbers are finite for a fixed base b. It would be interesting to investigate whether this is true for the other variations as well and under what conditions. Another open question is the asymptotic behavior of k^* and l^* as a function of b.

References

[1] R. S. Bird, Meertens number, *Journal of Functional Programming*, **8** (1998), 83–88.

[2] K. Gödel, Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I, *Monatshefte für Mathematik und Physik*, **38** (1931), 173–198.

[3] OEIS Foundation Inc., *Entry A246532 in the On-Line Encyclopedia of Integer Sequences*, (2022), `https://oeis.org/A246532`.

[4] B. Rosser, The n-th prime is greater than $n \log n$, *Proceedings of the London Mathematical Society*, **45** (1939), 21–44.

[5] J. B. Rosser and L. Schoenfeld, Approximate formulas for some functions of prime numbers, *Illinois Journal of Mathematics*, **6** (1962), 64–94.

[6] A. Turing, On computable numbers, with an application to the entscheidungsproblem, *Proceedings of the London Mathematical Society*, **42** (1937), 230–265.