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Big Two and n-Card Poker Probabilities

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Abstract

Between the poker hands of straight, flush, and full house, which hand is more common? In standard 5-card poker, the order from most common to least common is straight, flush, full house. The same order is true for 7-card poker such as Texas hold'em. However, is the same true for n-card poker for larger n? We study the probability of obtaining these various hands for n-card poker for various values of $n \geq 5$. In particular, we derive closed expressions for the probabilities of flush, straight and full house and show that the probability of a flush is less than a straight when $n \leq 11$, and is more than a straight when $n > 11$. Similarly, we show that the probability of a full house is less than a straight when $n \leq 19$, and is more than a straight when $n > 19$. This means that for games such as Big Two where the ordering of 13-card hands depends on the ordering in 5-card poker, the rank ordering does not follow the occurrence probability ordering, contrary to what intuition suggests.

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1 Introduction

In traditional games of chance, value is typically correlated with the rarity of the outcome. In 5-card poker [1], the hands are ranked by their likelihood of occurrence. The royal flush, of which there are only four hands, is ranked the highest as it is the most uncommon. However, Texas hold'em, a popular variant of poker, does not follow this trend. While the same 5-card hands are constructed, players have seven cards to choose from. This changes the probabilities such that the lowest hand, the high card hand, is rarer than the one pair and two pair hands [8]. The ranking of the hands is the same as in 5-card poker even though the probabilities have changed. This quirk of the game is an exception to the rule: that value is correlated with rarity, and these exceptions are more common than one might think.

In the card game Big Two [6] a deck of 52 cards is distributed among the players (generally four). From here on, we will assume that Big Two is played with four players with each player being dealt 13 cards. Cards are played in groups (e.g. single cards, pairs, five card hands, etc). The value of a five card hand follows the corresponding value in 5-card poker, however, the five card hand must be straight or higher. While a straight is a more common hand in 5-card poker than a full house or a flush, is this still true for 13-card poker? The 5-card poker hands of straight, flush and full house will be the focus of this paper as they are the most applicable to the card game Big Two.

The probability of obtaining specific hands in various variants of 5-card and 7-card poker are well studied [8]. For larger numbers of cards, such probabilities are less well known. Bill Butler [2] calculated probabilities for n-card poker up to $n = 10$ using a computer program implementing a brute-force approach to generate all possible hands. On the other hand, G. P. Michon provided a formula for the number of straight flush of n-card poker [3, 5]. In [4] the probabilities of a variant of poker with only 3 suits are considered.

However, general formulas for the number of flush, straight, and full house hands are not available for arbitrary n. The purpose of this paper is to give formulas and algorithms for computing the probabilities of flush, full house and straight in n-card poker and show that the relative order of these probabilities changes for different n.

2 Notations and Preliminaries

We assume a standard 52 card deck. The 4 suits are denoted as s_i for $i = 1, \cdots, 4$. The number of n-card hands is $\binom{52}{n}$. For a multiset R, let $\nu(R)$ denote the number of permutations of the *nonzero* elements of R given by a multinomial coefficient [7] and let $|R|$ denote the number of *nonzero* elements of R (counted with multiplicity).

We count the occurrences of each flush, straight, full house hand independently. For example if we have a hand with both a full house and a straight, we would count this as both types. If we were to only count it as a full house because it has a higher apriori value, that would influence the probabilities based on their apparent perceived rarity. The main goal of this paper is to determine the true probabilities of each hand without assuming apriori value rankings of each type of hand.

3 Number of flush hands

A flush is defined as a set of cards with 5 or more cards of the same suit. Let r_i denote the number of cards of suit s_i. Let S_n be the set of partitions of n into a multiset of 4 integers in the range $[0, 13]$. A hand is not a flush if and only if these integers are all less than 5. For each multiset S in S_n, the total number of hands whose number of cards in each suit corresponds to S can be computed as a product of 3 terms. The term $\binom{4}{|S|}$ describes the number of ways the multiset of nonzero elements of S can be arranged among the 4 suits as r_i's. The term $\nu(S)$ describes the number of arrangements of S, and finally $\Pi_i \binom{13}{r_i}$ describes the number of hands that has S as the number of cards of the same suit. Therefore, the number of non flush hands is equal to

$$\sum_{S\in S_n,\max(S)\leq 4} \binom{4}{|S|}\nu(S)\Pi_i\binom{13}{r_i}.$$

As an illustrative example, consider the case of $n = 7$ and $S = \{0, 2, 2, 3\}$, i.e. the 7-card hands that have 0 cards of a first suit, 2 cards of a second suit, 2 cards of a third suit and 3 cards of a fourth suit. Since $\max(S) = 3 < 5$, these hands do not contain a flush. The nonzero elements (or equivalently the zero elements) of S can be arranged in $\binom{4}{|S|} = 4$ ways among the 4 suits, there are $\nu(S) = 3$ permutations of the nonzero elements of S and given an assignment of S to r_i, there are $\binom{13}{2}\binom{13}{2}\binom{13}{3}$ ways to pick cards from these three suits making the number of hands corresponding to S equal to $12\binom{13}{2}\binom{13}{2}\binom{13}{3} = 20880288$.

The number of flush hands is then equal to

$$\binom{52}{n} - \sum_{S=\{r_1,r_2,r_3,r_4\}\in S_n, r_i\leq 4} \binom{4}{|S|}\nu(\{r_1, r_2, r_3, r_4\})\binom{13}{r_1}\binom{13}{r_2}\binom{13}{r_3}\binom{13}{r_4}.$$

It is clear that by the pigeonhole principle for $n \geq 17$ there will be more than 5 cards with the same suit and the probability of a flush is 100%. This remark shows that there are values of n such that the probability of flush is larger than the probability of straight as it is easy to find 17 cards with no straight hands.

4 Number of full house hands

A full house is defined as having 3 cards of the same rank and another 2 cards of the same rank. We can use the same analysis and formulas in Section 3 for flush hands by simply swapping rank with suit. In particular, let t_i denote the number of cards of rank i where $1 \leq i \leq 13$. Let H_n be the set of partitions of n into 13 integers in the range $[0, 4]$. Note that $|H_n| = |H_{52-n}|$ and $|H_n| \leq |H_{26}| = 104$, i.e. $n = 26$ is where the number of partitions is the largest. Let $T_n \subseteq H_n$ be the partitions in H_n such that the largest term is larger than or equal to 3 and the second largest term is larger than or equal to 2. Then the number of full house hands is equal to

$$\sum_{T\in T_n} \binom{13}{|T|}\nu(T)\Pi_i\binom{4}{t_i}.$$

As an example, consider $n = 7$ and $T = \{1, 2, 4, 0, \cdots, 0\}$, i.e. 7-card hands that have 1 card of a first rank, 2 cards of a second rank and 4 cards of a third rank and 0 cards of the other ranks. Note that $T \in T_7$ since the largest element of $T \geq 3$ and the second largest element of $T \geq 2$ and thus such hands contain a full house. $\nu(T) = 3$ and the number of full house hands of this configuration is $\binom{13}{3} \times 3 \times \binom{4}{1}\binom{4}{2}\binom{4}{4} = 20592$.

Again, by the pigeonhole principle, we can always find a full house when $n \geq 27$, since there will be two suits with all the rank drawn along with a card from the third suit and the probability will be 100%. This remark shows that there are values of n such that the probability of full house is larger than the probability of straight as it is easy to find 27 cards with no straight hands. This and the remark in the previous section show that when n is small (e.g. $n = 5$) a straight is more common than a full house or a flush, but less common when n is large. To get a clearer idea of when and how often this switch occurs, we need to derive a formula for the number of straight hands.

5 Number of straight hands

A straight is a list of 5 cards of sequential rank. We consider the standard high rules where the Ace can occur before Two and after a King card. The first card of a straight can be an Ace, Two, $\cdots$ all the way to a Ten card. Let t_i denote the number of cards of rank i and H_n be as defined in Section 4. Let $g(k) \leq \binom{13}{k}$ be the number of combinations of k distinct ranks (where each rank r is in $1 \leq r \leq 13$) which include a straight hand. Then, the number of straight hands is

$$\sum_{H \in H_n} g(|H|)\nu(H)\Pi_i \binom{4}{t_i}.$$

5.1 Finding $g(k)$

In order to prevent duplicate counting, let W_i^k be the set of k distinct ranks containing a straight where the smallest card in the straight has rank i. Therefore, $W_i^k \cap W_j^k = \emptyset$ if $i \neq j$. It is clear that $g(k) = 0$ for $k \leq 4$, since a straight hand needs 5 distinct ranks. $g(5) = 10$ since there are 10 ways that 5 cards of distinct ranks can be a straight hand, one for each of the 10 possible smallest cards. It is clear that $g(13) = 1$ as there is only one way to have 13 distinct ranks and a straight hand is in this arrangement. Similarly $g(12) = 13$ since each of the 13 arrangements of 12 distinct ranks contains a straight hand. $g(11) = 77$ since there are 78 arrangements of 11 distinct ranks and only one such arrangement does not contains a straight (when the ranks do not contain a 5 and a 10 card).

Next, we show that $g(6) = 71$. For 6 distinct ranks, $|W_1^6| = 8$, since the sixth card can be any card from rank 6 to rank 13. For $i > 1$, the sixth rank can be any card except rank $i - 1$, so $|W_i^6| = 7$. This implies that $g(6) = 8 + 9 \times 7 = 71$. Similarly to compute $g(7)$, $|W_1^7| = \binom{8}{2} = 28$, and for $i > 1$, $|W_i^7| = \binom{7}{2} = 21$, thus $g(7) = 28 + 9 \times 21 = 217$, and $g(8) = \binom{8}{3} + 9 \times \binom{7}{3} = 56 + 9 \times 35 = 371$. To compute $g(9)$, we continue to use the formula $\binom{8}{4} + 9 \times \binom{7}{4} = 70 + 9 \times 35 = 385$, but need to remove cases of double counting. This occurs for W_{10}^8 which contains an Ace in the straight hand and picking the remaining 4 cards to have rank $2, 3, 4, 5$ will intersect with W_1^8. Thus $g(9) = 385 - 1 = 384$.

Finally, for $g(10)$, we use the formula $\binom{8}{5} + 9 \times \binom{7}{5} = 56 + 9 \times 21 = 245$ and list below 11 cases of double counting. For W_{10}^{10}, the remaining cards being $(2,3,4,5,6)$, $(2,3,4,5,7)$, $(2,3,4,5,8)$, $(3,4,5,6,7)$, $(4,5,6,7,8)$ intersect with W_1^{10}, W_2^{10}, W_3^{10} or W_4^{10}. For W_9^{10}, the remaining cards being $(1,2,3,4,5)$, $(2,3,4,5,6)$, $(3,4,5,6,7)$ intersect with W_1^{10}, W_2^{10} and W_3^{10} respectively. For W_8, the remaining cards being $(1,2,3,4,5)$, $(2,3,4,5,6)$ intersect with W_1^{10}, and W_2^{10} respectively. For W_7^{10}, the remaining cards being $(1,2,3,4,5)$ intersects with W_1^{10}. Therefore $g(10) = 245 - 11 = 234$.

Thus we have shown that $g(k) = 0,0,0,0,10,71,217,371,384,234,77,13,1$ for $k = 1,\cdots,13$. These values have also been computed in [3] via a brute-force algorithm.

It is interesting to note that even though the 3 types of hands considered (straight, flush and full house) are very different, their analyses follow a similar approach and the formulas for the number of hands of each type are all of the form

$$\sum_{S\in S_n} g(|S|)\nu(S) \prod_{s_i\in S} \binom{K}{s_i};$$

a sum of products of 3 terms. These 3 terms have similar interpretations among the 3 type of hands. In the third term K is either 13 or 4 depending on whether the multiset S is about the count of the hand for each suit or rank respectively. The first term $g(|S|)$ describes the number of ways the multiset of the nonzero elements of S can be arranged among the suits (or ranks). The second term $\nu(S)$ describes the number of permutations of the nonzero elements of S and finally the third term $\prod_{s_i\in S}\binom{K}{s_i}$ describes the number of hands that have the same numbers of cards in each suit (or rank) as S.

5.2 An 11 card hand with no straight hand

It is easy to see that an 11 card hand of the same suit does not contain a straight hand if and only if the ranks missing are 5 and 10. On the other hand, a 12 card hand of the same suit must contain a straight hand. When $n \geq 45$, by the generalized pigeonhole principle there is a suit with 12 or more cards and thus the number of straight hands is $\binom{52}{n}$ and the probability of a straight hand is 100%. Similarly, when $n = 44$, the only possibility that there is not a suit with 12 or more cards is when all 4 suits have exactly 11 cards and the only configuration where there is not a straight is if all the suits are missing rank 5 and 10. This means that the number of straight hands in 44-card poker is $\binom{52}{44} - 1 = \binom{52}{8} - 1$.

6 Number of straight, flush and full house hands in n-card poker

The values described in Sections 3-5 are computed and shown in Table 1. Using the data in Table 1, Figure 1 shows the probability of occurrence of a straight, flush or full house in a random hand of n cards. It shows that flush is always more probable than a full house. We see that for $n < 12$, straight hands are more common that full hands and flush hands. There is a crossover point at $n = 12$, where for $n \geq 12$ flush hands are more common than straight hands. There is another crossover point at $n = 20$, where for $n \geq 20$ full house hands are more common than straight hands.

n	number of hands	number of straight	number of flush	number of full house
5	2598960	10240	5148	3744
6	20358520	367616	207636	166920
7	133784560	6454272	4089228	3514992
8	752538150	73870336	52406640	46541430
9	3679075400	619588736	491448100	435926920
10	15820024220	4051217344	3585287134	3087272188
11	60403728840	21461806976	21076866408	17297489352
12	206379406870	94674009184	102014990714	79387982102
13	635013559600	355161047872	412247470340	307061893424
14	1768966344600	1152374363488	1404025311000	1024024781208
15	4481381406320	3279045142912	4063219805320	2994831165040
16	10363194502115	8276491135968	10101843501490	7769077277923
17	21945588357420	18706297925768	21945588357420	18011190562092
18	42671977361650	38154873848572	42671977361650	37522889445106
19	76360380541900	70680929691448	76360380541900	70598500404172
20	125994627894135	119535302593662	125994627894135	120551073059703
21	191991813933920	185328058520744	191991813933920	187726126771040
22	270533919634160	264282641858276	270533919634160	267830920323824
23	352870329957600	347526172985064	352870329957600	351537171709152
24	426384982032100	422213549653051	426384982032100	425903913135844
25	477551179875952	474573239602540	477551179875952	477437987194480
26	495918532948104	493971477605994	495918532948104	495905472254088
27	477551179875952	476384056580348	477551179875952	477551179875952
28	426384982032100	425743257091369	426384982032100	426384982032100
29	352870329957600	352546752515104	352870329957600	352870329957600
30	270533919634160	270384470692560	270533919634160	270533919634160
31	191991813933920	191928737926752	191991813933920	191991813933920
32	125994627894135	125970392012595	125994627894135	125994627894135
33	76360380541900	76351947773568	76360380541900	76360380541900
34	42671977361650	42669338815712	42671977361650	42671977361650
35	21945588357420	21944852453408	21945588357420	21945588357420
36	10363194502115	10363013522632	10363194502115	10363194502115
37	4481381406320	4481342680672	4481381406320	4481381406320
38	1768966344600	1768959253568	1768966344600	1768966344600
39	635013559600	635012471952	635013559600	635013559600
40	206379406870	206379271078	206379406870	206379406870
41	60403728840	60403715596	60403728840	60403728840
42	15820024220	15820023274	15820024220	15820024220
43	3679075400	3679075356	3679075400	3679075400
44	752538150	752538149	752538150	752538150
45	133784560	133784560	133784560	133784560
46	20358520	20358520	20358520	20358520
47	2598960	2598960	2598960	2598960
48	270725	270725	270725	270725
49	22100	22100	22100	22100
50	1326	1326	1326	1326
51	52	52	52	52
52	1	1	1	1

Table 1: Number of hands in n-card poker.

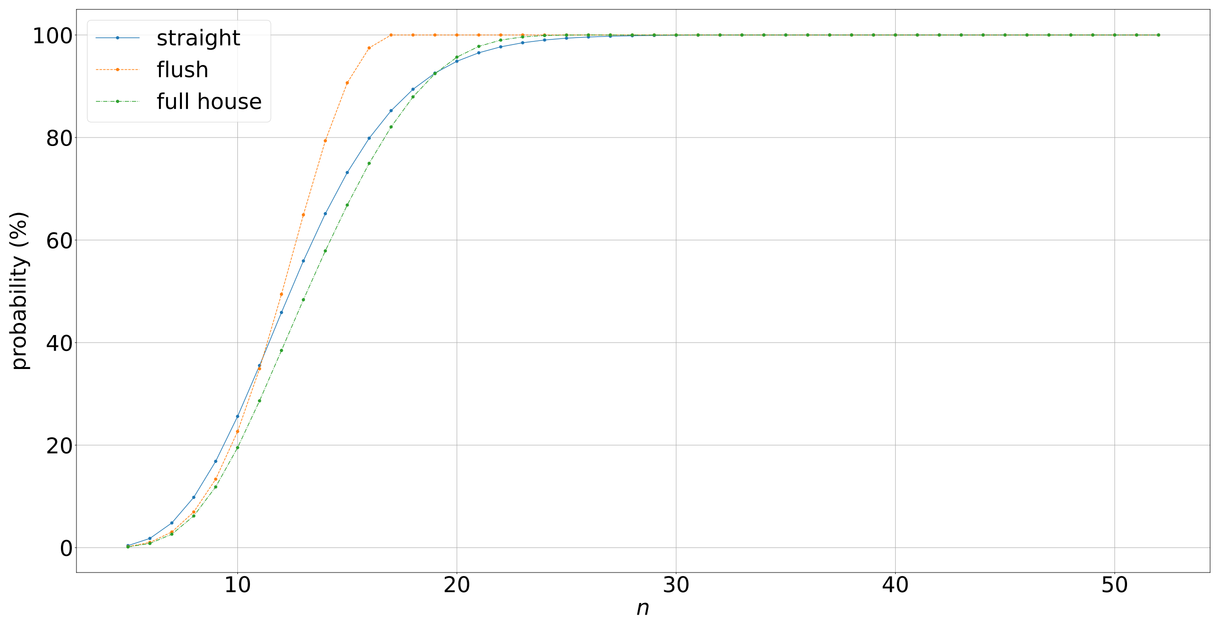

Figure 1: Probability of various hands in n-card poker.

6.1 Python code

In Listings 1-3 we provide the Python code used to generate the data in Table 1 and Figure 1.

```
# number of straight hands
from math import comb, prod, factorial
from sympy.utilities.iterables import partitions
def g(l):
    return (0,0,0,0,0,10,71,217,371,384,234,77,13,1)[l]
for n in range(5,53):
    k = 0
    for p in partitions(n,k=4,m=13):
        ps = []
        for d in p:
            ps.extend([d]*p[d])
        q = sorted(ps)
        k += g(len(ps)) \
            *(factorial(len(ps))//prod(factorial(q) \
            for q in p.values())) \
            *prod(comb(4,q) for q in ps)
    print(n,k)
```

Listing 1: Python code to compute the number of straight hands.

```
# number of flush hands
from math import comb, prod, factorial
from sympy.utilities.iterables import partitions
for n in range(5,53):
    k = 0
    for p in partitions(n,k=13,m=4):
        ps = []
        for d in p:
            ps.extend([d]*p[d])
        q = sorted(ps)
        if (max(q)<=4):
            k += comb(4,len(ps)) \
                *(factorial(len(ps))//prod(factorial(q) \
                for q in p.values())) \
                *prod(comb(13,q) for q in ps)
    print(n,comb(52,n)-k)
```

Listing 2: Python code to compute the number of flush hands.

```
# number of full house hands
from math import comb, prod, factorial
from sympy.utilities.iterables import partitions
for n in range(5,53):
    k = 0
    for p in partitions(n,k=4,m=13):
        ps = []
        for d in p:
            ps.extend([d]*p[d])
        q = sorted(ps)
        if (q[-1] >= 3 and q[-2] >= 2):
            k += comb(13,len(ps)) \
                *(factorial(len(ps))//prod(factorial(q) \
                for q in p.values())) \
                *prod(comb(4,q) for q in ps)
    print(n,k)
```

Listing 3: Python code to compute the number of full house hands.

7 Conclusions

We considered the poker hands of straight, flush, and full house in n-card poker. We show that a straight hand can be more or less probable than a flush or a full house depending on the value of n. This shows that in four player Big Two, the odds of drawing a hand with a flush are higher than the odds of drawing a hand with a straight, contrary to how they are scored.

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