Super Sudoku Squares

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Abstract

We determine the spectrum for a variant of a completed Sudoku puzzle.

I make the bold assumption that everyone is familiar with Sudoku puzzles. Here is a completed one:

1	6	8	5	7	3	9	2	4
2	4	9	6	8	1	7	3	5
3	5	7	4	9	2	8	1	6
4	9	2	8	1	6	3	5	7
5	7	3	9	2	4	1	6	8
6	8	1	7	3	5	2	4	9
7	3	5	2	4	9	6	8	1
8	1	6	3	5	7	4	9	2
9	2	4	1	6	8	5	7	3

Figure 1

We coordinatize this as follows. The rows are divided into three groups, called *fat rows*, (fr), labeled 1, 2, 3. Each fat row consists of three *skinny rows*, (sr), also labelled 1, 2, 3. Similarly, we have *fat columns*, (fc), and *skinny columns*, (sc), also similarly labelled. Each of the 81 occupied cells is assigned a label of 4 coordinates, $(\mathrm{fr}, \mathrm{sr}, \mathrm{fc}, \mathrm{sc})$. For example, in the above array, symbol 9 occurs in these 9 cells: $(1,1,3,1)$, $(1,2,1,3)$, $(1,3,2,2)$, $(2,1,1,2)$, $(2,2,2,1)$, $(2,3,3,3)$, $(3,1,2,3)$, $(3,2,3,2)$, $(3,3,1,1)$.

To be a valid Sudoku square, each row, and each column must contain each symbol exactly once. Furthermore, each *block* (= intersection of a fat row and a fat column) must contain each symbol exactly once. Here's the definition for other values of 3 (-:

Let N be a finite non-empty set of size $|N| = n$. (We took $N = \{1,2,3\}$ above.) A *Sudoku square* of *order* n, $\mathrm{SS}(n)$, is a function $f : N^4 \to S$, where S is a set of n^2 symbols, satisfying the following properties. (Here $f(c)$ is the symbol occupying cell c.) Each of the following functions is a bijection from N^2 to S: for all $i, j \in N$,

$$\begin{aligned}(x,y) &\to f(i,j,x,y),\\ (x,y) &\to f(x,y,i,j),\\ (x,y) &\to f(i,x,j,y).\end{aligned}$$

To simplify notation, we denote these three properties by $f(i,j,\_,\_)$, $f(\_,\_,i,j)$, $f(i,\_,j,\_)$ respectively.

We leave the proof of the following to the reader:

Theorem 1. *There is a* $\mathrm{SS}(n)$ *for every positive integer* n.

The three properties above are defined by choosing two of the four blanks in $f(\ ,\ ,\ ,\ )$ to be replaced by underscores _. But there are six such choices. Accordingly, we define a *Super Sudoku square* of *order* n, $\mathrm{SSS}(n)$, to be a function $f : N^4 \to S$, satisfying, for all $i, j \in N$, all six of the properties

$$\begin{aligned}&f(i,j,\_,\_),\\ &f(\_,\_,i,j),\\ &f(i,\_,j,\_),\\ &f(i,\_,\_,j),\\ &f(\_,i,j,\_),\\ &f(\_,i,\_,j).\end{aligned}$$

Do these exist? The SS(3) above is actually a SSS(3)! Let's check the last three properties at $i = 3$, $j = 2$.

	3			4			8	
	1			5			9	
	2			6			7	

Figure 2: $f(3, \_, \_, 2)$

			4	9	2			
			7	3	5			
			1	6	8			

Figure 3: $f(\_, 3, 2, \_)$

	5			9			1	
	8			3			4	
	2			6			7	

Figure 4: $f(\_, 3, \_, 2)$

Note that in all three cases, the nine occupied cells contain each of the nine symbols exactly once. Those are three of the twenty-seven properties; we leave the rest to the reader to check.

So for which positive integer n is there a SSS(n)? Try $n = 2$, and you'll fail. So 2 is out.

To investigate this question, we need an alternate definition of super sudoku squares. For this, we will take S to be $N \times N$ from now on, so our symbols are ordered pairs of elements of N. For a positive integer k, we say an n^k by k array is *complete* if its rows consist of the n^k vectors with k coordinates from N, each one occurring exactly once. We define A to be an n^4 by 6 *Super Sudoku array* of *order* n, SSA(n), if it's six columns are indexed by the set $\{\mathrm{fr}, \mathrm{sr}, \mathrm{fc}, \mathrm{sc}, \mathrm{s}, \mathrm{t}\}$, and the n^4 by 4 subarray consisting of the columns of A indexed by S is complete, for each of the 7 following sets S: $\{\mathrm{fr}, \mathrm{sr}, \mathrm{fc}, \mathrm{sc}\}$, $\{\mathrm{fr}, \mathrm{sr}, \mathrm{s}, \mathrm{t}\}$, $\{\mathrm{fc}, \mathrm{sc}, \mathrm{s}, \mathrm{t}\}$, $\{\mathrm{fr}, \mathrm{fc}, \mathrm{s}, \mathrm{t}\}$, $\{\mathrm{fr}, \mathrm{sc}, \mathrm{s}, \mathrm{t}\}$, $\{\mathrm{sr}, \mathrm{fc}, \mathrm{s}, \mathrm{t}\}$, $\{\mathrm{sr}, \mathrm{sc}, \mathrm{s}, \mathrm{t}\}$.

Theorem 2. *There is a* SSS(n) *if and only if there is a* SSA(n).

Proof. First, suppose there is a SSS(n). We construct a SSA(n) as follows: for all $(i, j, k, l) \in N^4$, if cell (i, j, k, l) contains symbol (a, b), then (i, j, k, l, a, b) is a row of the SSA. The condition imposed by the first set S listed above states that each cell of the SSA contains exactly one symbol. The other six sets S correspond to the six conditions listed above in the definition of SSS(n). For example, $S = \{\mathrm{fr}, \mathrm{sc}, \mathrm{s}, \mathrm{t}\}$ corresponds to condition $f(i, \_, \_, j)$. Thus an SSS(n) produces an SSA(n). But the construction is reversible, so also an SSA(n) produces an SSS(n). □

We'll need some background before proceeding; see [2] for more detail. A *Latin square* of *order* n, LS(n), is an array, with rows and columns indexed by N, filled by symbols from N, so that each row and each column contains each symbol exactly once. Thus a SS(n) is a LS(n^2). We say two such squares, A and B, are said to be orthogonal, if for all $(i, j) \in N^2$, there is a unique pair $(x, y) \in N^2$ so that cell (x, y) contains symbol i in square A, and symbol j in square B. Such a pair is known a POLS(n). Here is an example for $n = 3$.

3	1	2
1	2	3
2	3	1

1	3	2
2	1	3
3	2	1

Figure 5

POLSs were introduced by the great 18th century Swiss mathematician Leonhard Euler. He produced a POLS(n) for all n odd, and all n a multiple of 4. He observed that there is none with $n = 2$, and conjectured the same for any n congruent to 2 modulo 4. Nearly two centuries later, Bose, Shirkhande and Parker [1], proved there is a POLS(n) if and only if n is not 2 or 6.

Here is an alternate definition. An n^2 by 4 orthogonal array, which we here abbreviate to OA(n), is an array with 4 columns indexed by $C = \{\mathrm{r}, \mathrm{c}, \mathrm{s}, \mathrm{t}\}$, with the property that the n^2 by 2 subarray indexed by S is complete, for every two-element subset S of C.

Theorem 3. *There is a* POLS(n) *if and only if there is an* OA(n)*.*

Proof. Here's the correspondence: if (x, y, i, j) is a row of the OA(n), fill cell (x, y) with symbol i in the first square, and symbol j in the second square. We leave the details to the reader. □

As an example, the POLS(3) in Figure 5 give the following OA(3):

1	1	3	1
1	2	1	3
1	3	2	2
2	1	1	2
2	2	2	1
2	3	3	3
3	1	2	3
3	2	3	2
3	3	1	1

Figure 6

Theorem 4. *If there is a* SSS(n)*, then there is a* POLS(n)*.*

Proof. For an SSA(n) A, let $@ \in N$, and define an n^2 by 4 array B as follows: for each row $(w, x, y, z, @, @)$ of A, (w, x, y, z) is a row of B. It is straight forward to show B is an OA(n). □

As an example, the SSS(3) in Figure 1, gives the OA(3) in Figure 6, upon taking $(@, @)$ to be symbol 9.

While a SSA(n) contains $6n^4$ symbols, and a POLS(n) only $4n^2$ symbols, a construction in the other direction seems unlikely. And yet:

Theorem 5. *If there is a* POLS(n)*, then there is a* SSS(n)*.*

Proof. We take N to be $\mathbb{Z}_n$, the ring of integers modulo n. Let B be a POLS(n). We construct a SSA(n) A as follows: for each row (w, x, y, z) of B, and each $a, b \in N$, take $(w, x, y + a, z + b, a, b)$ to be a row of A.

We must show that the n^4 subarray of A with columns indexed by S is complete, for each of the seven subsets S listed above in the definition of SSA(n). The seven verifications are all similar; we content ourselves with the proof for $S = \{\text{fr}, \text{sc}, \text{s}, \text{t}\}$. We need to show that if $i, j, k, l \in N$, then there is a unique row (w, x, y, z) of B, and also a unique pair $(a, b) \in N^2$, so that $(w, x, y + a, z + b, a, b) = (i, p, q, j, k, l)$, for some $p, q \in N$. So a must equal k, and b must equal l. Also, we must have $w = i$, and $z + b = j$, whence $z = j–b = j–l$. But B is a POLS(n), so there are unique $x, y \in N$ so that $(w, x, y, z) = (i, x, y, j–l)$ is a row of B. □

Note 6. Of the fifteen four-element subsets S of $\{\text{fr}, \text{sr}, \text{fc}, \text{sc}, \text{s}, \text{t}\}$, the corresponding subarray of A is complete for thirteen of them, not just the seven required above.

Corollary 7. *If n is a positive integer, there is a* SSS(n) *if and only if n is not* 2 *or* 6.

Proof. This follows from the Bose-Shirkhande-Parker result cited above. □

References

[1] R. C. Bose, S. S. Shirkhande, and E. T. Parker, Further results on the construction of mutually orthogonal Latin squares and the falsity of Euler's conjecture, *Canadian J. Math.* **12** (1960), 189–203.

[2] C. C. Lindner and C. A. Rodger, *Design Theory* (2009), Chapman & Hall/CRC, Chapter 6.